INTRODUCTION TO THE
LAPLACE TRANSFORMATION

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An Introduction to the

LAPLACE TRANSFORMATION

With Engineering Applications

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WITH 31 DIAGRAMS

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PREFACE

THIS book contains the substance of a course of lectures delivered to engineers and physicists at the National Standards Laboratory, Sydney, in 1944, when the author was seconded to the Radiophysics Laboratory of the Council for Scientific and Industrial Research. A mimeographed edition was produced by the C.S.I.R. in 1945 for circulation in Australia.

There is an increasing demand for a knowledge of operational methods by engineers and physicists who encounter transient problems, but whose mathematical equipment is restricted to that of an ordinary engineering course. Of the many books available, the older ones use the rather suspect and certainly obscure " operational methods ", while the newer ones require a knowledge of the theory of functions of a complex variable, and some comparatively advanced mathematics. Here I have attempted to go as far as possible using no mathematics beyond ordinary calculus, but at the same time catering for the reader who wishes to acquire the manipulative skill necessary to solve his own problems : to this end, the book contains as little theory as possible ; it is, in fact, largely a collection of worked examples illustrating the methods of solution of the various types of problem commonly arising in circuit theory ; a handful of problems, with answers, has been added to enable the reader to test his skill.

For problems on ordinary linear differential equations, and thus on electric circuits with lumped constants, the use of the theory of functions of a complex variable is quite unnecessary ; but it is needed for a complete study of partial differential equations, and in the higher parts of the subject. The engineer who wishes to go deeply into these matters must study this theory, and since quite sophisticated points in it do arise, he must study it fully and carefully, and so before he starts he is entitled to

know what types of problem he will be able to solve, and how useful the solutions will be. I have therefore in Chapter IV discussed the equations of the transmission line from the point of view of the earlier chapters, using calculus only; this treatment covers all the common problems, and will enable the reader to follow published work on the subject. It is to be understood that solutions obtained in this way are correct, and that the algebra is the same as it would be if the complex variable were used : the latter merely changes the point of view, and allows a complete mathematical discussion of the validity of the solutions.

The reasons for the notation used here for the Laplace transform

$$\int_0^\infty e^{-pt} y(t) dt \qquad . \qquad . \qquad (1)$$

of $y(t)$ may be given briefly. Any notation should satisfy three conditions : (i) it must be short, and easy to write ; (ii) it must be applicable to any symbols ; (iii) it should not be used in any other connection. The simplest notation of all would be to write a capital letter Y for the Laplace transform of the corresponding small letter y ; but this may be most inconvenient since it is not applicable to many symbols, in particular to capital and many Greek letters. A " star ", y^*, is not much used in other connections but is rather clumsy to write. The " bar " notation, $\bar{y}$, used here, was originally used by Lévy and reintroduced by Carslaw ; it is easy to write, and can be applied to any symbols, but it has the disadvantage of being employed in other senses, these, however, rarely occur in those fields where the Laplace transformation is most useful, and ease of writing is a most important desideratum. If preferred it can be replaced by $\tilde{y}$ used by Millman.

A much more difficult question is that of the relative merits of the form (1) and of the " p-multiplied " form

$$p \int^\infty e^{-pt} y(t) dt \qquad . \qquad . \qquad .. \qquad (2)$$

Both notations have their own minor advantages, but the essential point is that form (2) gives an exact correspondence with the large body of work done by the older operational methods, while (1) corresponds with the newer books, with most of the theoretical work, and with corresponding work on the Fourier transform.

Finally, it is a pleasure to acknowledge the assistance given to me by many officers of the C.S.I.R., in particular that of the Secretary in making available the drawings for the text-figures.

REFERENCES

The following are recommended for further reading and are occasionally referred to below. They all use the complex variable. The first four use the form (1) of the Laplace transform, and the last three the form (2).

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CONTENTS

FUNDAMENTAL THEORY

1. *The Laplace transform*

In this book we shall primarily be concerned with problems on the behaviour of dynamical or electrical systems which are started in a given way at some instant, which we take to be the zero of time, $t = 0$. Mathematically, these are called " initial value problems ", and may be expressed as the solution of a system of differential equations with given initial values at the instant $t = 0$. Negative values of the time do not arise in either the specification of the problems or in their solution. The Laplace transformation is a mathematical device which is useful for solving such problems : if a function is specified for all positive values of the time, we can (in the simple cases needed here) write down a related quantity called its Laplace transform ; conversely, if the Laplace transform is known, the value of the function can be found from it. The importance of the method arises from the fact that, for the types of problem we are interested in, it is easy to write down the Laplace transform of the solution, and from this the solution itself is then easily found.

We suppose $y(t)$ to be a known function of t for values of $t > 0$. Then the Laplace transform $\bar{y}(p)$ of $y(t)$ is defined as

$$\bar{y}(p) = \int_0^\infty e^{-pt} y(t) dt, \qquad . \qquad . \quad (1.1)$$

where p is a number * sufficiently large to make the integral (1.1) convergent. For example, if $y(t) = 1$ any positive

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* Of course p may be a complex number whose real part is sufficiently large to make (1.1) convergent, but, in the early parts of the theory, it is a little more definite to think of p as a real positive number, and nothing is lost by this restriction. The Laplace transform is one of a number of " integral transforms " formed as in (1.1) by multiplying a function of t by a chosen

p is permissible, but if $y(t) = e^{2t}$ we must have $p > 2$. Apart from this p is unrestricted, and $\bar{y}(p)$ is a function of p just as $y(t)$ is of t. We shall occasionally write them in full in this way to emphasize this dependence, but usually shall write $\bar{y}$ for the Laplace transform of y, it being understood that these are functions of p and t respectively.

We need first a collection of the Laplace transforms of a few common functions. For example, if $y = e^{at}$, where a may be real or complex,

$$\bar{y} = \int_0^\infty e^{-pt} \cdot e^{at}dt = \int_0^\infty e^{-(p-a)t}dt = \frac{1}{p-a}, \quad (1.2)$$

where p must be greater than the real part of a.

Again, if a is real and

$$y = \sin at = \frac{1}{2i}(e^{iat} - e^{-iat}),$$

using (1.2) we have

$$\bar{y} = \frac{1}{2i}\left(\frac{1}{p-ia} - \frac{1}{p+ia}\right) = \frac{a}{p^2+a^2}. \quad (1.3)$$

Proceeding in this way, or quoting the results from known definite integrals, we can build up the Table I of Laplace transforms which contains *all* the transforms needed in the first three chapters. This Table, and Theorems I to III below, provide the whole of the theory needed in Chapters I and II.

In (1.6), as remarked above, p has to be greater than the real part of a, while in (1.7) to (1.12) a is real, and in (1.9) and (1.10) p must be greater than * $|a|$. All the results of Table I are elementary integrals except (1.11) and (1.12), which have been included because they arise in cases of resonance. The simplest way of deriving

function of t and p, and integrating with respect to t. The Fourier transform (26.1), is another example. Each integral transform has characteristic properties, but those of the Laplace transform which are proved below make it the most suitable for the solution of initial value problems.

* If a is real, $|a| = a$, if $a > 0$; $|a| = -a$, if $a < 0$.

results of this type is as follows : written out in full (1.8) states that

$$\int_0^\infty e^{-pt} \cos at\, dt = \frac{p}{p^2 + a^2}.$$

TABLE I

$y(t)$	$\bar{y}(p)$	
1	$\dfrac{1}{p}$	$\cdot$ $\cdot$ $\cdot$ $\cdot$ (1.4)
$\dfrac{t^{n-1}}{(n-1)!}$	$\dfrac{1}{p^n}$, n a positive integer	(1.5)
e^{at}	$\dfrac{1}{p-a}$ $\cdot$ $\cdot$ $\cdot$ (1.6)	
$\sin at$	$\dfrac{a}{p^2 + a^2}$ $\cdot$ $\cdot$ $\cdot$ (1.7)	
$\cos at$	$\dfrac{p}{p^2 + a^2}$ $\cdot$ $\cdot$ $\cdot$ (1.8)	
$\sinh at$	$\dfrac{a}{p^2 - a^2}$ $\cdot$ $\cdot$ $\cdot$ (1.9)	
$\cosh at$	$\dfrac{p}{p^2 - a^2}$ $\cdot$ $\cdot$ $\cdot$ (1.10)	
$\dfrac{t}{2a}\sin at$	$\dfrac{p}{(p^2 + a^2)^2}$ $\cdot$ $\cdot$ $\cdot$ (1.11)	
$\dfrac{1}{2a^3}(\sin at - at\cos at)$	$\dfrac{1}{(p^2 + a^2)^2}$ $\cdot$ $\cdot$ $\cdot$ (1.12)	

If we differentiate * both sides of this with respect to a, as if a were a variable, we get

$$\int_0^\infty e^{-pt} t \sin at\, dt = \frac{2ap}{(p^2 + a^2)^2},$$

* This technique of differentiating under the integral sign with respect to a parameter is a powerful weapon of pure mathematics ; naturally its use needs justification, and this can usually be supplied when, as here, it is employed in deducing new Laplace transforms from known ones.

which is equivalent to (1.11). Treating (1.7) in the same way gives (1.12).

The collection of transforms in Table I can be greatly extended by the use of the following Theorem.

THEOREM I.   *If $\bar{y}(p)$ is the Laplace transform of $y(t)$, and a is any number, real or complex, then $\bar{y}(p + a)$ is the Laplace transform of $e^{-at}y(t)$.*

This follows immediately, since the Laplace transform of $e^{-at}y(t)$ is

$$\int_0^\infty e^{-pt} \cdot e^{-at}y(t)dt = \int_0^\infty e^{-(p+a)t}y(t)dt = \bar{y}(p + a).$$

As an example of the use of this Theorem, we see from (1.5) that the transform of

$$\frac{e^{-at}\,t^{n-1}}{(n-1)!} \text{ is } \frac{1}{(p+a)^n}.$$

Similarly from Theorem I and (1.8) it follows that the transform of

$$e^{-bt} \cos at \text{ is } \frac{p+b}{(p+b)^2 + a^2},$$

and from Theorem I and (1.7) that the transform of

$$e^{-bt} \sin at \text{ is } \frac{a}{(p+b)^2 + a^2}.$$

2. *To find the function $y(t)$ which has a given Laplace transform.*

It will be seen in §4 that the Laplace transformation method of solving an ordinary linear differential equation with constant coefficients and given initial conditions consists of writing down an algebraic equation for the transform $\bar{y}(p)$ of the solution $y(t)$. This transform will usually be found to have the form

$$\frac{f(p)}{g(p)}, \qquad . \qquad . \qquad . \qquad . \qquad (2.1)$$

where $f(p)$ and $g(p)$ are polynomials in p which have no common factor, the degree of $f(p)$ being lower than that of $g(p)$.

Most of the labour * of the solution consists of finding the $y(t)$ corresponding to a $\bar{y}(p)$ given in this form. This is done by looking up $\bar{y}(p)$ in the Table I of transforms, possibly combined with Theorem I. It is shown at the end of this secton that the $y(t)$ found in this way is the only function which has $\bar{y}(p)$ for transform.

For example, if

$$\bar{y} = \frac{1}{p+2}, \text{ it follows from (1.6) that } y = e^{-2t}.$$

Similarly, from (1.7) and (1.8), if

$$\bar{y} = \frac{2p+6}{p^2+4}, y = 2 \cos 2t + 3 \sin 2t.$$

If the denominator of $\bar{y}$ is a general quadratic in p, we complete the square, and use Theorem I and the Table. For example, if

$$\bar{y} = \frac{p+7}{p^2+2p+5} = \frac{p+7}{(p+1)^2+4} = \frac{p+1}{(p+1)^2+4} + \frac{6}{(p+1)^2+4},$$

it follows from Theorem I and the Table that

$$y = e^{-t} \cos 2t + 3e^{-t} \sin 2t.$$

Notice that *both* numerator and denominator of $\bar{y}$ have to be expressed in terms of $p + a$ (in the case above, $p + 1$) before Theorem I is used.

If the denominator of $\bar{y}$ is of higher degree than the second, we have first to find its factors,† then we express $\bar{y}$ in partial fractions by the usual algebraic methods, and finally write down the function of t corresponding to each fraction using Table I. Alternatively, Theorem II below may be used when it is applicable.

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* The student is particularly advised to make himself skilful in the processes of this section. Labour can be saved by the use of a more extensive table of transforms than that given here, but transforms not included in the tables at present available frequently arise, and in any case a solid background of manipulative skill is extremely useful.

† For the method of doing this in problems where they are not obvious see § 12.

EXAMPLE 1. $\bar{y} = \dfrac{p + 1}{p(p^2 + 4p + 8)}$.

Assume * $\dfrac{p + 1}{p(p^2 + 4p + 8)} = \dfrac{A}{p} + \dfrac{Bp + C}{p^2 + 4p + 8}$.

Then $p + 1 \equiv A(p^2 + 4p + 8) + p(Bp + C)$.   (2.2)

Equating coefficients of p^2, p, and the constant term, respectively, we have

$$A + B = 0,$$
$$4A + C = 1,$$
$$8A = 1.$$

Solving these for A, B, and C, we find

$$\bar{y} = \frac{1}{8p} - \frac{p - 4}{8(p^2 + 4p + 8)} = \frac{1}{8p} - \frac{(p + 2) - 6}{8[(p + 2)^2 + 4]}. \quad (2.3)$$

Then from Theorem I and the Table, we find

$$y = \tfrac{1}{8} - \tfrac{1}{8}e^{-2t}\{\cos 2t - 3 \sin 2t\}.$$

The process of equating coefficients used above is always available, but the results can often be obtained more quickly otherwise. For example, if we put $p = 0$ in the identity (2.2) we find $A = 1/8$. Then, putting $A = 1/8$ in (2.2), this becomes $Bp + C \equiv \tfrac{1}{8}(4 - p)$, and we get the result (2.3) obtained before.

EXAMPLE 2. $\bar{y} = \dfrac{1}{(p + 1)^2(p^2 + 4)}$.

Assume $\dfrac{1}{(p + 1)^2(p^2 + 4)} = \dfrac{A}{p + 1} + \dfrac{B}{(p + 1)^2} + \dfrac{Cp + D}{p^2 + 4}$.

Then $1 \equiv A(p+1)(p^2+4) + B(p^2+4) + (Cp+D)(p+1)^2$.

Equating coefficients we find

$$A = \tfrac{2}{25}, \ B = \tfrac{1}{5}, \ C = -\tfrac{2}{25}, \ D = -\tfrac{3}{25}.$$

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* It is shown in text-books on algebra that if we assume a fraction with a constant numerator corresponding to each linear factor of the denominator, and a fraction with a linear numerator corresponding to each quadratic factor (or squared linear factor) of the denominator, and so on, we shall always have just the right number of equations to determine the unknown constants in these fractions.

Then, using the Table and Theorem I, we get

$$y = \tfrac{2}{25}e^{-t} + \tfrac{1}{5}te^{-t} - \tfrac{2}{25}\cos 2t - \tfrac{3}{50}\sin 2t.$$

There is a simple general Theorem applicable to many important cases :

THEOREM II. *If $\bar{y}(p) = f(p)/g(p)$, where $f(p)$ and $g(p)$ are polynomials in p, the degree of $f(p)$ being less than that of $g(p)$, and if*

$$g(p) = (p - a_1)(p - a_2) \ldots (p - a_n), \quad (2.4)$$

where $a_1, a_2, \ldots, a_n$ are constants, which may be real or complex but must all be different, then

$$\bar{y}(p) = \sum_{r=1}^{n} \frac{f(a_r)}{(p - a_r)g'(a_r)} \qquad (2.5)$$

$$= \sum_{r=1}^{n} \frac{1}{p - a_r} \cdot \frac{f(a_r)}{(a_r - a_1) \ldots (a_r - a_{r-1})(a_r + a_{r-1}) \ldots (a_r - a_n)} \quad (2.6)$$

$$= \sum_{r=1}^{n} \frac{1}{p - a_r} \cdot \left[\frac{(p - a_r)f(p)}{g(p)} \right]_{p=a_r}. \qquad (2.7)$$

Also the function $y(t)$ whose Laplace transform is $\bar{y}(p)$ is

$$y(t) = \sum_{r=1}^{n} \frac{f(a_r)}{g'(a_r)} e^{a_r t} \qquad (2.8)$$

$$= \sum_{r=1}^{n} \frac{f(a_r) e^{a_r t}}{(a_r - a_1) \ldots (a_r - a_{r-1})(a_r - a_{r+1}) \ldots (a_r - a_n)} \quad (2.9)$$

$$= \sum_{r=1}^{n} \left[\frac{(p - a_r)f(p)}{g(p)} \right]_{p=a_r} e^{a_r t}. \qquad (2.10)$$

(2.8) is the form taken by *Heaviside's Expansion Theorem* in the present method.* Since (2.8) . . . (2.10) follow directly from (2.5) . . . (2.7), respectively, by

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* In Heaviside's notation, and in the " p-multiplied " form of the Laplace transformation, in which (in order to have an exact correspondence with the Heaviside notation) the Laplace transform is defined as p times the integral in (1.1), an additional term appears arising from this extra factor, p, and the simplicity of (2.8) is lost.

using (1.6), it appears that the essential content of Theorem II is simply the formulæ (2.5) . . . (2.7) for putting $f(p)/g(p)$ into partial fractions in the case in which $g(p)$ has no repeated factors. This is a well-known result,* but for completeness we give a proof below.

All of (2.5) to (2.10) are useful; (2.6) is easily remembered by noticing that the coefficient of $1/(p-a_r)$ is the result of putting $p = a_r$ in $f(p)/g(p)$, *omitting* the factor $(p-a_r)$ of $g(p)$; (2.7) is simply another way of writing this statement. Similar remarks apply to (2.9) and (2.10). The partial fraction formulæ (2.5) and (2.6) are themselves very useful, cf. Example 4, below. With regard to (2.8) and (2.9), it is usually better to use (2.9) when $g(p)$ has a few simple factors, as in the examples of this section. It sometimes happens that $g(p)$ has complicated factors, which need not even be set out explicitly, as in the problems of § 16, and then (2.8) is the more useful.

To derive (2.6) we put $\bar{y}(p)$ into partial fractions by the methods discussed earlier, by assuming

$$\frac{f(p)}{g(p)} = \sum_{r=1}^{n} \frac{A_r}{p - a_r}. \qquad . \qquad . \quad (2.11)$$

Then

$$f(p) \equiv \sum_{r=1}^{n} A_r (p-a_1) \ldots (p-a_{r-1})(p-a_{r+1}) \ldots (p-a_n).$$

Putting $p = a_r$ in this gives

$$f(a_r) = A_r(a_r-a_1) \ldots (a_r-a_{r-1})(a_r-a_{r+1}) \ldots (a_r-a_n),$$
$$r = 1, 2, \ldots, n,$$

and substituting this value in (2.11) gives (2.6). To deduce (2.5), we notice that differentiating (2.4) with respect to p by the ordinary rule for differentiating a product gives

$$g'(p) = \sum_{r=1}^{n} (p-a_1) \ldots (p-a_{r-1})(p-a_{r+1}) \ldots (p-a_n).$$

Putting $p = a_r$ in this, gives

$$g'(a_r) = (a_r-a_1) \ldots (a_r-a_{r-1})(a_r-a_{r+1}) \ldots (a_r-a_n). \quad (2.12)$$

Using (2.12) in (2.6) gives (2.5).

* Cf. Gibson, *Treatise on the Calculus*, edn. 2 (1906), § 120.

Before proceeding further it is desirable to settle some questions of terminology. Firstly, suppose $\bar{y}(p)$ is given in the form (2.1). The simplest case is that of (2.4) in which $g(p)$ is a product of n different factors,

$$g(p) = (p - a_1)(p - a_2) \ldots (p - a_n).$$

The numbers $a_1, a_2, \ldots, a_n$ are the *roots* of the equation $g(p) = 0$. They may also be spoken of as the *zeros* of $g(p)$, since they are the values of p for which $g(p)$ is zero. Finally, they are the values of p for which the function $\bar{y}(p)$ is infinite, and in this sense are described as the *poles* of $\bar{y}(p)$. In the more general case in which $g(p)$ has factors such as $(p - a_1)^n$, a_1 is called a multiple (or more precisely an *n*-ple) root of $g(p) = 0$, an *n*-ple zero of $g(p)$, or a pole of order n of $\bar{y}(p)$.

Secondly, there is the question of the notation for a complex number. If

$$z = x + iy$$

is a complex number, we call x the real part of z, y the imaginary part of z, and $x - iy$ the conjugate of z. The number may be represented on the Argand diagram of Fig. 1, and may also be specified by its modulus, $|z| = \sqrt{(x^2 + y^2)}$, and the angle ϕ defined * by any *two* of
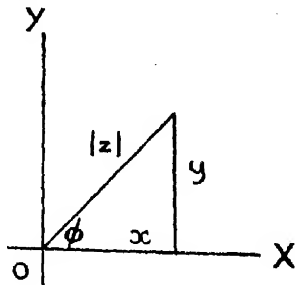
FIG. 1.

$$\sin \phi = \frac{y}{|z|}, \quad \cos \phi = \frac{x}{|z|}, \quad \tan \phi = \frac{y}{x} \ldots \quad (2\cdot13)$$

We shall call this angle ϕ the argument of z, written arg z; other names for it are the amplitude of z, (amp z), or the phase of z.

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* This definition still leaves ϕ undetermined by an integral multiple of 2π; this does not matter in the applications made here. If desired, ϕ can be restricted to the range $\pi > \phi > -\pi$, when it is called the Principal Value of arg z.

With the notation (2.13), any complex number $x + iy$ may be written in the form

$$x + iy = \sqrt{(x^2 + y^2)}e^{i\phi}, \qquad . \qquad . \qquad (2.14)$$

where
$$\phi = \arg(x + iy).$$

The importance of this in the present connection is that when $y(t)$ is found from $\bar{y}(p)$ by the use of (2.9), it will frequently involve products or quotients of complex numbers, and the most rapid way of reducing such expressions to a real form is by using (2.14) for each of the complex numbers (cf. Example 5, below).

One pitfall of some importance must be mentioned: the definition of ϕ in (2·13) is equivalent to *

$$\phi = \arg(x + iy) = \tan^{-1}(y/x), \qquad \text{if } x > 0 \\ = \pi + \tan^{-1}(y/x), \text{ if } x < 0 \Big\}, \qquad (2\cdot15)$$

thus, if we had written $\phi = \tan^{-1}(y/x)$ in (2·14) (and this is very frequently done), there might be an error of π in the argument of $x + iy$, leading to an error in sign in the final result. This point is particularly important in Tables of transforms, and in problems such as Example 5 below, where numerical values of some parameters are not specified and thus may be taken with either sign.

We now give some examples on the use of Theorem II.

EXAMPLE 3. $\bar{y} = \dfrac{1}{p(p + 1)(p + 2)(p + 3)}.$

By (2.6),

$$\bar{y} = \frac{1}{1 . 2 . 3 . p} + \frac{1}{(-1) . 1 . 2 . (p+1)}$$
$$+ \frac{1}{(-2)(-1) . 1 . (p+2)} + \frac{1}{(-3)(-2)(-1)(p+3)},$$

and y follows from (1.6). Or, using (2.9),

$$y = \frac{1}{1 . 2 . 3} + \frac{1}{(-1) . 1 . 2}e^{-t} + \frac{1}{(-2)(-1) . 1}e^{-2t}$$
$$+ \frac{1}{(-3)(-2)(-1)}e^{-3t}.$$

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* $\tan^{-1} x$ is defined as the angle between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose tangent is x.

EXAMPLE 4. $\bar{y} = \dfrac{1}{p^2(p^2 + 1)(p^2 + 4)}$.

Using (2.6) on $\bar{y}$, *regarded as a function of* p^2, we find

$$\bar{y} = \frac{1}{4p^2} - \frac{1}{3(p^2 + 1)} + \frac{1}{12(p^2 + 4)}.$$

Thus
$$y = \frac{1}{4}t - \frac{1}{3}\sin t + \frac{1}{24}\sin 2t.$$

Problems on resistanceless circuits often lead to transforms which can be conveniently treated, as above, as functions of p^2.

EXAMPLE 5.

$$\bar{y} = \frac{p + a}{(p + b)[(p + \mu)^2 + n^2]}$$
$$= \frac{p + a}{(p + b)(p + \mu - in)(p + \mu + in)}. \quad (2\cdot16)$$

This is a generalization of Example 1, and could be dealt with *either* by resolving into partial fractions with real denominators, as was done in Example 1, *or* by resolving into partial fractions with linear complex denominators, that is, by using one of the forms of Theorem II. Thus applying (2.9) to (2.16) gives *

$$y = \frac{(a - b)}{(\mu - b)^2 + n^2}e^{-bt}$$
$$+ \left\{\frac{a - \mu + in}{(b - \mu + in)2in}e^{-\mu t + int} + \text{Conjugate}\right\} \quad (2.17)$$

Then writing
$$\phi_1 = \arg[(a - \mu) + in]$$
$$\phi_2 = \arg[(b - \mu) + in],$$

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* A pair of conjugate roots of $g(p) = 0$ give in (2.9) the sum of a pair of conjugate terms, of which only one need be written down. The method used below for reducing such a pair of terms to real form will be found very convenient.

and using (2.14) in the complex numbers in (2.17), we find

$$y = \frac{(a-b)}{(\mu-b)^2+n^2}e^{-bt}$$
$$+ \frac{1}{n}\left\{\frac{(a-\mu)^2+n^2}{(b-\mu)^2+n^2}\right\}^{\frac{1}{2}}e^{-\mu t}\left\{\frac{e^{i(nt+\phi_1-\phi_2)}}{2i} + \text{Conjugate}\right\}$$

$$= \frac{(a-b)}{(\mu-b)^2+n^2}e^{-bt}$$
$$+ \frac{1}{n}\left\{\frac{(a-\mu)^2+n^2}{(b-\mu)^2+n^2}\right\}^{\frac{1}{2}}e^{-\mu t}\sin(nt+\phi_1-\phi_2).$$

Theorem II covers the case * *in which $g(p)$ has no repeated factors ;* for example, it is not applicable to Examples 2 or 4 above. It is easy to generalize the theorem to the case of repeated factors. Theorem II may be stated in the form that, *to each linear factor $(p-a_r)$ of the denominator of $\bar{y}(p)$ there corresponds a term*

$$\frac{f(a_r)}{g'(a_r)}e^{a_r t} \qquad . \qquad . \qquad . \quad (2.18)$$

in the solution.

The generalization is that, *to each squared factor $(p-b)^2$ of the denominator of $\bar{y}(p)$ there corresponds a term*

$$\left[\frac{(p-b)^2 f(p)}{g(p)}\right]_{p=b} te^{bt} + \left[\frac{d}{dp}\left\{\frac{(p-b)^2 f(p)}{g(p)}\right\}\right]_{p=b} e^{bt} \quad (2.19)$$

in the solution.

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* In (2.1) it was stated that $f(p)$ and $g(p)$ were to have no common factors : this is desirable but not essential. With complicated expressions (such as those of § 16) it would greatly increase the labour to separate out such a factor, and, in fact, if $f(p)$ and $g(p)$ both have the same factor $(p-a)$, the term in e^{at} corresponding to this in Theorem II has zero coefficient, and so the application of Theorem II gives the correct result. If $g(p)$ contained a factor $(p-a)^2$, and $f(p)$ a factor $(p-a)$, Theorem II could be applied after cancelling the factor $(p-a)$, whereas a cursory inspection would suggest that it is not applicable and that (2.18) would have to be used (this, also, would give the correct result).

And to each m-ple factor $(p - c)^m$ of the denominator of $\bar{y}(p)$ there corresponds a term

$$\sum_{s=0}^{m-1} \left[\frac{d^s}{dp^s} \left\{ \frac{(p - c)^m f(p)}{g(p)} \right\} \right]_{p=c} \frac{t^{m-s-1}}{s\,!\,(m - s - 1)\,!} \, e^{ct} \quad (2.20)$$

in the solution.

As an example, consider the problem of Example 2 above, in which

$$\bar{y} = \frac{1}{(p + 1)^2(p^2 + 4)} = \frac{1}{(p + 1)^2(p - 2i)(p + 2i)}.$$

Then by (2.19) and (2.18)

$$y = \left[\frac{1}{p^2 + 4} \right]_{p=-1} te^{-t} + \left[\frac{d}{dp} \left\{ \frac{1}{p^2 + 4} \right\} \right]_{p=-1} e^{-t}$$
$$+ \left\{ \frac{1}{4i(1 + 2i)^2} e^{2it} + \text{Conjugate} \right\},$$

leading to the same result as before.

Finally, the question of "uniqueness" must be discussed. To a given function $y(t)$ there corresponds a single transform $\bar{y}(p)$ defined by (1.1). It is not clear, however, that to a given transform $\bar{y}(p)$ there corresponds only a single function $y(t)$. Given $\bar{y}(p)$ we have sometimes been able to find *a* function $y(t)$ which has $\bar{y}(p)$ for transform by looking up $\bar{y}(p)$ in the Table of transforms ; but there is so far no reason why the function of t found this way should be the *only* function which has $\bar{y}(p)$ for transform. The same question arises in the integral calculus when we seek a function which has a given function for differential coefficient, and there it is found that there are infinitely many such functions (differing by additive constants).

In the present case the matter is settled definitely by *Lerch's Theorem,** which states that, *if two continuous*

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* For a proof see C. and J., Appendix I. The actual theorem is a great deal wider than that stated above, and includes also the useful result that if two functions $y_1(t)$ and $y_2(t)$ are continuous except at a number of ordinary discontinuities (i.e. jumps in value), and have the same Laplace transform, they must have the same points of discontinuity and can differ only at these points.

*functions have the same Laplace transform they must be
identical.* Thus, if we find a continuous function $y(t)$
from a given $\bar{y}(p)$, for example by the processes described
above which use the Table of transforms, this is the only
continuous function which has $\bar{y}(p)$ for transform.

It follows also from Lerch's Theorem that, if k is a
constant, and

$$\bar{y}(p) = 0, \ p > k, \ \text{then} \ y(t) = 0, \ t > 0 \ . \quad (2.21)$$

provided $y(t)$ is continuous ; and if $y(t)$ is not continuous
it can only differ from zero at isolated points.

3. Some additional theorems

THEOREM III.   *If $\bar{y}(p)$ is the Laplace transform * of
$y(t)$, and $y(t)$ is continuous † and tends to y_0 as $t \to 0$, then
the Laplace transform of dy/dt is $p\bar{y}(p) - y_0$.* (3.1)
*If, in addition, dy/dt is continuous and tends to y_1 as $t \to 0$,
then*

$$\text{the Laplace transform of } \frac{d^2y}{dt^2} \text{ is } p^2\bar{y}(p) - py_0 - y_1. \quad (3.2)$$

*And if $d^3y/dt^3 \ldots d^ny/dt^n$ are continuous and $d^ry/dt^r \to y_r$,
$r = 2, 3, \ldots, n - 1$, as $t \to 0$, then*

$$\text{the Laplace transform of } \frac{d^ny}{dt^n}$$

is $p^n\bar{y}(p) - p^{n-1}y_0 - \ldots - py_{n-2} - y_{n-1}.$ (3.3)

These results follow immediately by integration by
parts.   To prove (3.1) we have

$$\int_0^\infty e^{-pt} \frac{dy}{dt} \, dt = \left[ye^{-pt} \right]_0^\infty + p \int_0^\infty y e^{-pt} dt = -y_0 + p\bar{y},$$

---

* It is assumed that all the functions concerned have Laplace
transforms.   There are functions of t, such as e^{t^2}, for which the
integral (1.1) does not converge however large p is chosen, and
these will not have Laplace transforms.   Such functions are
not likely to appear in the types of problem to which the theory
is usually applied.   If they do, solutions may often be obtained
by the use of § 24, Theorem IX.

† If y is discontinuous, that is, has jumps in value, at certain
points, extra terms arise corresponding to these points.

since $ye^{-pt} \to y_0$ as $t \to 0$, and $ye^{-pt} \to 0$ as $t \to \infty$, provided p is sufficiently large.

Similarly, to prove (3.2), we have

$$\int_0^\infty e^{-pt}\frac{d^2y}{dt^2}\,dt = \left[\frac{dy}{dt}e^{-pt}\right]_0^\infty + p\int_0^\infty e^{-pt}\frac{dy}{dt}\,dt$$
$$= -y_1 - py_0 + p^2\bar{y}.$$

Repeating the process n times gives (3.3).

THEOREM IV. *If* $\bar{y}(p)$ *is the Laplace transform of* $y(t)$, *then*

$$\text{the transform of } \int_0^t y(t')dt' \text{ is } \frac{1}{p}\bar{y}(p). \qquad . \quad (3.4)$$

For

$$\int_0^\infty e^{-pt}dt\int_0^t y(t')dt' = \left[-\frac{1}{p}e^{-pt}\int_0^t y(t')dt'\right]_0^\infty + \frac{1}{p}\int_0^\infty e^{-pt}y(t)dt$$
$$= \frac{1}{p}\bar{y}(p),$$

since the integrated part vanishes at both limits.

The results of Theorems III and IV can often be used to obtain new transforms : as a simple example it follows from (1.7) and Theorem IV that

$$\frac{a}{p(p^2+a^2)} \text{ is the transform of } \int_0^t \sin at\,dt = \frac{1}{a}(1 - \cos at).$$

THEOREM V. *If* $\bar{y}(p)$ *is the Laplace transform of* $y(t)$, *and* ω *is a positive constant, then*

$$\frac{1}{\omega}\bar{y}\left(\frac{p}{\omega}\right) \text{ is the transform of } y(\omega t). \qquad . \quad (3.5)$$

For $\displaystyle\int_0^\infty e^{-pt}y(\omega t)dt = \frac{1}{\omega}\int_0^\infty e^{-(p/\omega)t'}y(t')dt' = \frac{1}{\omega}\bar{y}\left(\frac{p}{\omega}\right).$

This simple result is useful when it is desired to arrange the algebra of a problem in dimensionless form. For an example of its use see § 14.

4. *The solution of ordinary linear differential equations with constant coefficients.*

Writing $D^n y$ for $d^n y/dt^n$, the problem is to solve the differential equation

$$D^n y + c_1 D^{n-1} y + \ldots + c_{n-1} Dy + c_n y = f(t), \; t > 0, \quad (4.1)$$

where $c_1, \ldots, c_n$ are constants, $f(t)$ is a given function of t, and $y, Dy, \ldots, D^{n-1}y$ are to take the given values $y_0, y_1, \ldots, y_{n-1}$ when $t = 0$.

We multiply (4.1) by e^{-pt} and integrate with respect to t from 0 to ∞. This gives

$$\int_0^\infty e^{-pt} \{ D^n y + c_1 D^{n-1} y + \ldots + c_n y \} \, dt = \bar{f}(p), \quad (4.2)$$

where $\bar{f}(p)$ is the Laplace transform of $f(t)$, which can be written down * from Table I and Theorem I for the common types of function which usually occur.

On the left-hand side of (4.2) we use Theorem III, which gives

$$p^n \bar{y} - (p^{n-1} y_0 + p^{n-2} y_1 + \ldots + y_{n-1})$$
$$+ c_1 p^{n-1} \bar{y} - c_1 (p^{n-2} y_0 + p^{n-3} y_1 + \ldots + y_{n-2})$$
$$+ \ldots + c_{n-1} p \bar{y} - c_{n-1} y_0 + c_n \bar{y} \qquad = \bar{f}(p).$$

Or $\quad (p^n + c_1 p^{n-1} + c_2 p^{n-2} + \ldots + c_n) \bar{y}$

$$= \bar{f}(p) + c_{n-1} y_0 + c_{n-2}(p y_0 + y_1) + \ldots$$
$$+ (p^{n-1} y_0 + p^{n-2} y_1 + \ldots + y_{n-1}). \quad (4.3)$$

The equation (4.3) is called the *subsidiary equation* corresponding to the differential equation (4.1) and the given initial conditions. It is formed from the differential equation (4.1) by replacing $D^r y$, $r = 0, \ldots, n$, on the left-hand side, by $p^r \bar{y}$; on the right-hand side, $f(t)$ is replaced by its transform $\bar{f}(p)$, and terms containing the initial conditions are added according to the rule:

for Dy we add y_0
for $D^2 y$,, $p y_0 + y_1$
for $D^3 y$,, $p^2 y_0 + p y_1 + y_2$
for $D^r y$,, $p^{r-1} y_0 + p^{r-2} y_1 + \ldots + y_{r-1}.$

* The solution for a function whose Laplace transform is not known, or even which has no Laplace transform, can be obtained by the use of Theorem IX, § 24.

The subsidiary equation (4.3) gives $\bar{y}(p)$, the transform of the solution $y(t)$ of the problem. To find $y(t)$ from $\bar{y}(p)$, the methods of § 2 are used. Since, by the uniqueness theorem of § 2, only one continuous function $y(t)$ corresponds to a given $\bar{y}(p)$, it follows that this will be the unique solution of the problem.

EXAMPLE 1. $(D^2 + 1)y = 0,\ t > 0$

with $y = y_0,\ Dy = y_1$, when $t = 0$.

The subsidiary equation is

$$(p^2 + 1)\bar{y} = p y_0 + y_1.$$

Thus $$\bar{y} = \frac{p y_0 + y_1}{p^2 + 1}.$$

And $y = y_0 \cos t + y_1 \sin t$.

EXAMPLE 2. $(D^2 + 4D + 8)y = 1,\ t > 0$

with $y = 0,\ Dy = 1$, when $t = 0$.

Since the transform of 1 is $1/p$, the subsidiary equation is

$$(p^2 + 4p + 8)\bar{y} = \frac{1}{p} + 1.$$

That is, $$\bar{y} = \frac{p + 1}{p(p^2 + 4p + 8)}.$$

The function y corresponding to this has been found in § 2, Ex. 1, and for a similar but more general case in § 2, Ex. 5.

EXAMPLE 3. $(D + 1)^2 y = \sin 2t,\ t > 0$,

with y and Dy zero when $t = 0$.

The subsidiary equation is

$$(p + 1)^2 \bar{y} = \frac{2}{p^2 + 4}.$$

The function y corresponding to this $\bar{y}$ has been found in § 2, Ex. 2.

EXAMPLE 4. $(D^2 + n^2)y = \cos nt,\ t > 0$,

with $y = y_0,\ Dy = y_1$, when $t = 0$.

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The subsidiary equation is

$$(p^2 + n^2)\bar{y} = \frac{p}{p^2 + n^2} + py_0 + y_1.$$

Thus $\bar{y} = \dfrac{p}{(p^2 + n^2)^2} + \dfrac{py_0 + y_1}{p^2 + n^2}.$

Using (1.11), (1.7), (1.8), it follows that

$$y = \frac{t}{2n} \sin nt + y_0 \cos nt + \frac{y_1}{n} \sin nt.$$

EXAMPLE 5. $(D + 1)y = t^2 e^{-t}, \ t > 0,$

with $y = y_0,$ when $t = 0.$

Using (1.5) and Theorem I, the transform of $t^2 e^{-t}$ is found to be $2/(p + 1)^3$. Thus the subsidiary equation is

$$(p + 1)\bar{y} = \frac{2}{(p + 1)^3} + y_0,$$

i.e. $\bar{y} = \dfrac{2}{(p + 1)^4} + \dfrac{y_0}{p + 1}.$

Then, using (1.5) and Theorem I, we find

$$y = \tfrac{1}{3} t^3 e^{-t} + y_0 e^{-t}.$$

5. Simultaneous ordinary linear differential equations with constant coefficients.

Here, proceeding as in § 4, using the transformation procedure on each of the equations in turn, gives a system of algebraic equations, the subsidiary equations, for the transforms of the solutions. Suppose, for example, it is required to solve a system of n second order equations

$$\sum_{s=1}^{n} (a_{rs}D^2 + b_{rs}D + c_{rs})y_s = f_r(t),$$
$$r = 1, 2, \ldots, n, \, t > 0, \quad (5.1)$$

where the $f_r(t)$ are given functions of t, and y_r and Dy_r, $r = 1, \ldots, n$, are to take the values u_r and v_r when $t = 0$.

Multiplying the equations (5.1) by e^{-pt}, integrating with respect to t from 0 to ∞, and using (3.1) and (3.2) gives the system of n subsidiary equations

$$\sum_{s=1}^{n} (a_{rs}p^2 + b_{rs}p + c_{rs})\bar{y}_s = \bar{f}_r(p)$$
$$+ \sum_{s=1}^{n} [a_{rs}(pu_s + v_s) + b_{rs}u_s], \quad r = 1, \ldots, n. \quad (5.2)$$

These have to be solved for $\bar{y}_1, \ldots, \bar{y}_n$, and then $y_1, \ldots, y_n$ are determined as in § 2.

EXAMPLE. $\left. \begin{array}{l} (D^2 + 2)x - Dy = 1 \\ Dx + (D^2 + 2)y = 0 \end{array} \right\} t > 0$

to be solved with $x = x_0$, $Dx = y = Dy = 0$, when $t = 0$.
The subsidiary equations are

$$(p^2 + 2)\bar{x} - p\bar{y} = \frac{1}{p} + px_0$$

$$p\bar{x} + (p^2 + 2)\bar{y} = x_0.$$

Solving we find

$$\bar{x} = \frac{p^4 x_0 + p^2(3x_0 + 1) + 2}{p(p^2 + 1)(p^2 + 4)} = \frac{1}{2p} + \frac{p(2x_0 - 1)}{3(p^2 + 1)} + \frac{p(2x_0 - 1)}{6(p^2 + 4)}$$

$$\bar{y} = \frac{2x_0 - 1}{(p^2 + 1)(p^2 + 4)} = \frac{2x_0 - 1}{3}\left(\frac{1}{p^2 + 1} - \frac{1}{p^2 + 4}\right).$$

Thus $x = \frac{1}{2} + \frac{1}{3}(2x_0 - 1) \cos t + \frac{1}{6}(2x_0 - 1) \cos 2t$
   $y = \frac{1}{3}(2x_0 - 1) \sin t - \frac{1}{6}(2x_0 - 1) \sin 2t.$

6. *Comparison with the ordinary method of solving differential equations, and with Heaviside's operational calculus.*

The ordinary method of solving the linear differential equation of order n,

$$\phi(D)y \equiv (D^n + c_1 D^{n-1} + \ldots + c_n)y = f(t), \quad (6.1)$$

consists of finding the complementary function

$$\sum_{r=1}^{n} A_r e^{a_r t}, \qquad . \qquad . \qquad . \quad (6.2)$$

where $A_1, \ldots, A_n$ are arbitrary constants, and the $a_1, \ldots, a_n$ are the roots of $\phi(p) = 0$ (provided these are all different, if they are not the complementary function takes a slightly different form provided for by the theory)

and adding to this a particular integral determined from $\phi(\mathrm{D})$ and $f(t)$ by certain rules. This gives the general solution of (6.1). To find the solution satisfying given initial conditions, it is necessary to solve n algebraic equations for the constants $A_1, \ldots, A_n$.

In the Laplace transformation method, the subsidiary equation (4.3) is

$$\bar{y} = \frac{1}{\phi(p)}\{\bar{f}(p) + \text{terms involving the initial conditions}\}. \quad (6.3)$$

To determine y from (6.3) we have to put the right-hand side into partial fractions,* and for this the roots $a_1, \ldots, a_n$ of $\phi(p) = 0$ are needed; thus these must be calculated in either method, but when they have been found, the work by the Laplace transformation method is much shorter; for example, if $a_1, \ldots, a_n$ are all different, y can be *written down* by (2.8), while with the older method of solution the n algebraic equations for the constants $A_1, \ldots, A_n$ of (6.2) still have to be solved. Thus the advantage of the Laplace transform method over the ordinary one increases enormously as the order of the equation to be solved increases.

The same remark applies to simultaneous linear differential equations—the greater the number and order of these, the greater is the advantage of the Laplace transformation method.

The part of $y(t)$ derived from the term $\bar{f}(p)/\phi(p)$ of (6.3) is a particular integral of (6.1), but it may differ from the particular integral found by the rules of the older theory by terms of the complementary function.

To see the connection between the present method and Heaviside's operational method, we again discuss (6.1). Heaviside usually considered only the case in which the initial conditions were zero, and $f(t)$ was his " unit function ", which is zero for $t < 0$ and unity for $t > 0$,

* If $f(t)$ is not one of the simple functions dealt with hitherto, it may not be possible to put $\bar{f}(p)/\phi(p)$ into partial fractions, but the solution can always be found as in § 24.

and is represented by the symbol **1**. On the left-hand side of (6.1), D is replaced by p; this is regarded as an operator (not as a number as in the Laplace transformation method). Thus (6.1) leads to

$$\phi(p)y = \mathbf{1}$$

or
$$y = \frac{1}{\phi(p)} \cdot \mathbf{1}, \qquad . \qquad . \qquad . \quad (6.4)$$

which is the operational expression for the solution y. The solution itself is then derived from (6.4) by Heaviside's expansion theorem which is similar in type to (2.8).

For this problem, the Laplace transform of the solution is

$$\bar{y} = \frac{1}{p\phi(p)}, \qquad . \qquad . \qquad . \quad (6.5)$$

and this differs from (6.4) only by the extra factor p in the denominator. In all cases the Heaviside operational expression is just p times the corresponding Laplace transform.

As remarked in § 1, the Laplace transformation procedure is designed for, and applies immediately to, the "initial value problems" of dynamics and electric circuit theory, in which differential equations have to be solved for times $t > 0$ with given initial conditions at $t = 0$. In other fields, such as the theory of elasticity, "boundary value problems" arise, in which a differential equation has to be solved in a region, say $0 < x < l$, and the solution has to satisfy some conditions at the boundary $x = 0$, and some at the boundary $x = l$. The older method of solution, as the sum of a complementary function (6.2) and a particular integral, applies equally well to both types, since, in a boundary value problem, there will still be n conditions, some at $x = 0$, and some at $x = l$, to determine the $A_1, \ldots, A_n$. The Laplace transformation can still be used, since the initial values $y_1, \ldots, y_n$ may be regarded as arbitrary constants; some of these may be fixed by the conditions at $x = 0$, and the remainder can be determined from the conditions at $x = l$

As an example, we consider the deflection * of a uniform, uniformly loaded, beam of length l, freely hinged at the origin, and clamped horizontally at the other end, $x = l$. Writing D for differentiation with respect to x, the differential equation for the deflection is

$$\text{EID}^4 y = w, \quad 0 < x < l, \qquad . \qquad . \quad (6.6)$$

where E, I, and w are constants. This has to be solved with $x = \text{D}^2 x = 0$ at $x = 0$, and $x = \text{D}x = 0$ at $x = l$. We solve (6.6) with values $0, y_1, 0, y_3$ of y, Dy, D$^2 y$, D$^3 y$, respectively, at $x = 0$, regarding y_1 and y_3 as unknown constants to be determined from the conditions at $x = l$. Writing $\bar{y}$ for the Laplace transform of y with respect to the present independent variable x, the subsidiary equation is

$$p^4 \bar{y} = \frac{w}{\text{EI}p} + p^2 y_1 + y_3.$$

Thus $y = \dfrac{wx^4}{24\text{EI}} + \dfrac{y_3 x^3}{6} + y_1 x.$

The conditions $y = \text{D}y = 0$ at $x = l$, give

$$\frac{wl^3}{24\text{EI}} + \frac{y_3 l^2}{6} + y_1 = 0,$$

$$\frac{wl^3}{6\text{EI}} + \frac{y_3 l^2}{2} + y_1 = 0.$$

Thus $y_1 = \dfrac{wl^3}{48\text{EI}}, \ y_3 = -\dfrac{3wl}{8\text{EI}},$

and $y = \dfrac{w}{48\text{EI}}(2x^4 - 3lx^3 + l^3 x).$

7. *The justification of the solutions of §§ 4, 5 from the pure-mathematical point of view.*

In deriving the subsidiary equations of §§ 4 and 5 some assumptions were made, which, from the point of view

* This method is well suited to problems of this type, particularly with concentrated or variable loads. For other examples see § 21 ; Jaeger, *Math. Gazette*, **23** (1939), 62 ; Pipes, *Journ App. Phys.*, **14** (1943), 486.

of strict pure mathematics, would have to be defended. For example, it was assumed that y and its derivatives had Laplace transforms. To make the solution quite rigorous further discussion would be needed, but this can be avoided by checking the results obtained by the methods above, that is to say verifying that they do satisfy the differential equations and initial conditions of their problems. This can be done quite generally, and since the process itself is purely algebraical and adds nothing to the theory of the subject it is sufficient to state the results here.

For the problem of § 4, the ordinary linear differential equation of order n with constant coefficients, it can be verified * that the solution obtained by the Laplace transformation method does satisfy the differential equation and initial conditions.

For the problem of § 5, simultaneous ordinary linear differential equations, it is possible to give a similar general verification that the solutions obtained by the Laplace transformation method satisfy the differential equations, and that they satisfy the initial conditions, *provided* that the determinant of the coefficients of the highest powers of D in the terms does not vanish. This restriction is required, not because of a flaw in the method of solution, but because of a fault in the specification of the problem. To see what it implies, consider the system of equations (5.1): in this case the determinant in question is

$$\begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}$$

If this vanishes it is possible, by adding together appropriate constant multiples of the equations (5.1), to form at least one equation containing no second derivatives. Thus, instead of having a system of n second order equations, which could be solved with $2n$ initial conditions, we have in fact a system of $n - 1$ (or less) second order equations,

* C. and J., §§ 34, 35 ; Doetsch, *loc. cit.*, Kap. 18.

and some first order or algebraic equations, and for such a system only $2n - 1$, or less, arbitrary initial conditions can be prescribed. If the problem is stated with $2n$ initial conditions, there must be relations between these or there will not be a solution.

Problems of this type occasionally arise in practical applications, also similar difficulties occur in switching problems in circuit theory. These questions are discussed in § 20.

EXAMPLES ON CHAPTER I

1. Verify the following Laplace transforms :

$\bar{y}(p)$	$y(t)$
$\dfrac{1}{(p-a)(p-b)}$	$\dfrac{1}{(a-b)}(e^{at} - e^{bt})$
$\dfrac{p}{(p-a)(p-b)}$	$\dfrac{1}{(a-b)}(ae^{at} - be^{bt})$
$\dfrac{1}{p(p^2+a^2)}$	$\dfrac{1}{a^2}(1 - \cos at)$
$\dfrac{1}{p^2(p^2+a^2)}$	$\dfrac{1}{a^3}(at - \sin at)$
$\dfrac{1}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{1}{ab(b^2-a^2)}(b \sin at - a \sin bt)$
$\dfrac{p}{(p^2+a^2)(p^2+b^2)}$	$\dfrac{1}{b^2-a^2}(\cos at - \cos bt)$
$\dfrac{1}{p[(p+\alpha)^2+\beta^2]}$	$\dfrac{1}{\alpha^2+\beta^2} - \dfrac{e^{-\alpha t}}{\beta\sqrt{(\alpha^2+\beta^2)}}$ $\sin[\beta t + \arg(\alpha + i\beta)]$.

2. Solve the following differential equations with initial values $y_0, y_1, \ldots$, of y, Dy, etc.

(i) $(D + \mu)y = 1$. $\left[y = \dfrac{1}{\mu} + \left(y_0 - \dfrac{1}{\mu} \right)e^{-\mu t} \right].$

(ii) $(D^2 + 5D + 6)y = 1$.
$$[y = \tfrac{1}{6} + (3y_0 + y_1 - \tfrac{1}{2})e^{-2t} + (\tfrac{1}{3} - y_1 - 2y_0)e^{-3t}].$$

(iii) $(D^2 + n^2)y = \sin \omega t.$

$$\left[y = \frac{\omega + y_1(\omega^2 - n^2)}{n(\omega^2 - n^2)} \sin nt + y_0 \cos nt - \frac{1}{\omega^2 - n^2} \sin \omega t\right].$$

(iv) $(D + 1)^2 y = \sin t.$

$$[y = (\tfrac{1}{2} + y_0 + y_1)te^{-t} + (\tfrac{1}{2} + y_0)e^{-t} - \tfrac{1}{2} \cos t].$$

3. If $\qquad d^4y/dx^4 = x, \quad 0 < x < l,$

with $\qquad y = dy/dx = 0$ at $x = 0$ and $x = l,$

show that $\qquad y = \dfrac{1}{120}x^2(x^3 + 2l^3 - 3xl^2).$

series with a leaky condenser of capacity C and leakage conductance $1/G$, the subsidiary equations are

$$\left(Lp + R + \frac{1}{G + Cp}\right)I = \bar{V} + L\overset{\circ}{I} - \frac{\overset{\circ}{Q}}{G + Cp} \quad (8.9)$$

$$\bar{Q} = \frac{C(\bar{I} + \overset{\circ}{Q})}{G + Cp}. \quad . \quad . \quad (8.10)$$

Returning to (8.6), we remark that it involves the initial current in the inductance and the initial charge on the condenser (these are related to the initial kinetic and potential energies, respectively, of the current). Further, since the terms involving them appear in the right-hand side added to the transform $\bar{V}$ of the applied voltage, it follows that the effect of initial charge $\overset{\circ}{Q}$ on the condenser is the same as that of an applied voltage whose transform is $(-\overset{\circ}{Q}/Cp)$, that is of a constant voltage $(-\overset{\circ}{Q}/C)$; similarly, the effect of initial current $\overset{\circ}{I}$ in the inductance is the same as that of an applied voltage whose transform is $L\overset{\circ}{I}$, this will be found in § 19 to be an impulsive voltage $L\overset{\circ}{I}\delta(t)$. The same remark is true in general—that initial conditions may be treated as voltage sources of the appropriate types.

9. *Examples on L, R, C circuits.*

EXAMPLE 1.   *Constant voltage, E, applied at $t = 0$ to L, R, C in series,* * *with zero initial current and charge.*

Here the subsidiary equation (8.6) is

$$\left(Lp + R + \frac{1}{Cp}\right)I = \frac{E}{p}. \quad . \quad . \quad (9.1)$$

Thus,

$$I = \frac{E}{L(p^2 + (R/L)p + (1/LC))} = \frac{E}{L[(p+\mu)^2+n^2]}, \quad (9.2)$$

where

$$\mu = \frac{R}{2L}, \; n^2 = \frac{1}{LC} - \mu^2 = \frac{1}{LC} - \frac{R^2}{4L^2} \quad (9.3)$$

The notation (9.3) will frequently be used later.

* For the exceptional case $L = R = 0$ and related problems, see § 20.

From (9.2), using Theorem I and (1.7), (1.5), and (1.9), respectively, we find

$$I = \frac{E}{nL}e^{-\mu t}\sin nt, \quad \text{if } n^2 > 0$$

$$= \frac{E}{L}te^{-\mu t}, \quad \text{if } n^2 = 0$$

$$= \frac{E}{kL}e^{-\mu t}\sinh kt, \quad \text{if } n^2 = -k^2 < 0.$$

EXAMPLE 2. *A condenser of capacity C charged to voltage E is discharged at $t = 0$ through an inductive resistance L, R.*

Here $\mathring{Q} = CE$, $\mathring{I} = 0$, $V = 0$, and (8.6) becomes

$$\left(Lp + R + \frac{1}{Cp}\right)I = -\frac{E}{p}, \quad . \quad . \quad (9.4)$$

which is the same as (9.1) except for the negative sign. This implies that the current starts flowing away from the positive side of the condenser.

Using (8.5) and (9.4), the transform of the charge on the condenser is found to be

$$\bar{Q} = \frac{CE}{p} - \frac{CE}{p(LCp^2 + RCp + 1)} = \frac{CE(p + 2\mu)}{(p + \mu)^2 + n^2},$$

where the notation (9.3) has been used. Then, in the case $n^2 > 0$, we have

$$Q = CEe^{-\mu t}\left\{\cos nt + \frac{\mu}{n}\sin nt\right\}.$$

EXAMPLE 3. *Steady current E/R is flowing from a battery of voltage, E, through a parallel combination of condenser C and inductive resistance L, R (Fig. 3), when at $t = 0$ the switch S is opened. It is required to find the potential across the condenser terminals.*

After the switch S has been opened, the circuit consists of L, R, C in series. The initial charge $\mathring{Q}$ on the condenser

is CE, and the initial current $\overset{\circ}{\text{I}}$ in the inductance is $-\,\text{E}/\text{R}$, the minus sign being chosen, since, from the conditions of the problem, this is flowing *away* from the positive side of the condenser. With these initial values (8.6) becomes

$$\left(\text{L}p + \text{R} + \frac{\text{I}}{\text{C}p}\right)\text{I} = -\frac{\text{LE}}{\text{R}} - \frac{\text{E}}{p}.$$

Thus

$$\bar{\text{I}} = -\frac{\text{E}(p + 2\mu)}{\text{R}[(p + \mu)^2 + n^2]},$$

where the notation (9.3) has been used. Hence, considering only the case $n^2 > 0$,

$$\text{I} = -\frac{\text{E}}{\text{R}}e^{-\mu t}\left(\cos nt + \frac{\mu}{n}\sin nt\right).$$
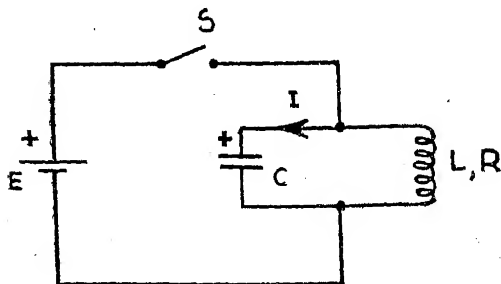
FIG. 3.

If $v = \text{Q}/\text{C}$ is the voltage across the condenser terminals, it follows from (8.5) that

$$\bar{v} = \frac{\bar{\text{Q}}}{\text{C}} = \frac{\text{E}}{p} - \frac{\text{E}(\text{L}p + \text{R})}{\text{R}p(\text{LC}p^2 + \text{RC}p + \text{I})}$$
$$= \frac{\text{E}[p + (\text{R}/\text{L}) - (\text{I}/\text{RC})]}{(p + \mu)^2 + n^2},$$

$$v = \text{E}e^{-\mu t}\left\{\cos nt + \frac{\text{I}}{n}\left(\frac{\text{R}}{2\text{L}} - \frac{\text{I}}{\text{RC}}\right)\sin nt\right\}.$$

This is the simplest type of circuit breaker restriking problem in which it is required to find the voltage across the contacts of a switch after they have been opened. In this case the voltage across the switch S is $E - v$. Another method of solving this problem is given in Ex. 2 at the end of this Chapter.

EXAMPLE 4. *Alternating voltage E sin ($\omega t + \phi$) applied at t = 0 to an L, R, C circuit with zero initial current and charge.*

Since $\sin (\omega t + \phi) = \sin \omega t \cos \phi + \cos \omega t \sin \phi$, its transform is

$$\frac{p \sin \phi + \omega \cos \phi}{p^2 + \omega^2}. \qquad . \qquad . \quad (9.5)$$

Thus, using the notation (9.3), the subsidiary equation (8.6) gives in this case

$$I = \frac{E}{L} \cdot \frac{p^2 \sin \phi + \omega p \cos \phi}{(p^2 + \omega^2)[(p + \mu)^2 + n^2]}. \qquad . \quad (9.6)$$

We consider only the most interesting case $n^2 > 0$. Here to find I we may either put (9.6) into partial fractions with real quadratic denominators, or notice that its denominator is

$$(p - i\omega)(p + i\omega)(p + \mu - in)(p + \mu + in),$$

and use (2.9). The latter method gives *

$$I = \frac{E}{L} \left\{ \frac{- \omega^2 \sin \phi + i\omega^2 \cos \phi}{2i\omega[(\mu + i\omega)^2 + n^2]} e^{i\omega t} + \text{Conjugate} \right\}$$

$$+ \frac{E}{L} \left\{ \frac{(- \mu + in)^2 \sin \phi + \omega(- \mu + in) \cos \phi}{2in[(- \mu + in)^2 + \omega^2]} e^{-\mu t + int} \right.$$

$$\left. + \text{Conjugate} \right\} \quad (9.7)$$

---

* Notice that if only the steady state part of I is required, only the terms of (2.9) arising from the zeros $\pm i\omega$ of the denominator of (9.6) need be written down. The other zeros $- \mu \pm in$ give transient terms. See also § 13.

To reduce this to real form, we use (2.14) as in § 2, Ex. 5. Thus, writing

$$\frac{L}{\omega}[2\mu\omega + i(\omega^2 - \mu^2 - n^2)] = Ze^{i\delta}, \qquad . \qquad (9.8)$$

$$(\mu^2 + \omega^2 - n^2) + 2i\mu n = Z_1 e^{i\delta_1},$$

$$[(\mu^2 - n^2)\sin\phi - \mu\omega\cos\phi] + i[n\omega\cos\phi - 2\mu n\sin\phi] = Z_2 e^{i\delta_2},$$

(9.7) becomes

$$I = \frac{E}{Z}\sin(\omega t + \phi - \delta) + \frac{EZ_2}{nLZ_1}e^{-\mu t}\sin(nt + \delta_1 + \delta_2), \qquad (9.9)$$

where Z and δ may be written out explicitly as the modulus and argument of the complex number on the left of (9.8), and similarly Z_1, δ_1, Z_2, δ_2.

10. *Electrical networks*

It is convenient to regard a complicated electrical network as built up of a number of circuit elements of the
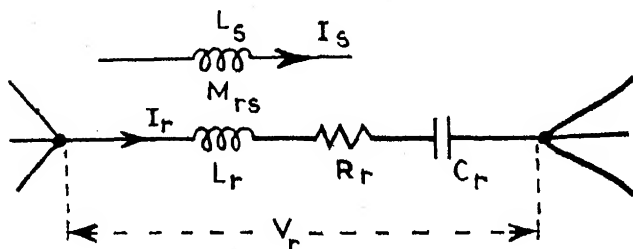
Fig. 4.

type studied in § 8, with their terminals connected in some specified manner. Let L_r, R_r, C_r be the inductance resistance, and capacity in the r-th of these elements (Fig. 4), let V_r be the voltage drop over it, let I_r be the current in the element, with initial value $\dot{I}_r$, and let Q be the charge on the condenser, with initial value $\dot{Q}_r$. To be quite general mutual inductance must be included

so let M_{rs} be the mutual inductance * between the r-th and s-th elements.

Then the equations for the r-th circuit element are

$$L_r\frac{dI_r}{dt} + R_rI_r + \frac{Q_r}{C_r} + \sum_{s \neq r} M_{rs}\frac{dI_s}{dt} = V_r, \quad (10.1)$$

$$\frac{dQ_r}{dt} = I_r. \quad (10.2)$$

Proceeding as in § 8, the subsidiary equation for the r-th circuit element is found to be

$$z_r\bar{I}_r + \sum_{s \neq r} z_{rs}\bar{I}_s = \bar{V}_r + L\mathring{I}_r + \sum_{s \neq r} M_{rs}\mathring{I}_s - \frac{\mathring{Q}_r}{C_r p}, \quad (10.3)$$

where

$$z_r = L_r p + R_r + \frac{1}{C_r p}, \quad . \quad . \quad (10.4)$$

$$z_{rs} = M_{rs} p. \quad . \quad . \quad . \quad (10.5)$$

From the form of the network, further equations can be written down connecting the I_r and V_r in different circuit elements. Firstly we have Kirchhoff's First Law †

$$\sum I = 0, \text{ at a junction.} \quad . \quad . \quad (10.6)$$

This leads to

$$\sum \bar{I} = 0, \text{ at a junction.} \quad . \quad . \quad (10.7)$$

Kirchhoff's Second Law states that, for any closed circuit in the network, the sum of the externally applied voltages, V, in the closed circuit, equals the sum of the voltage drops, V_r, over the circuit elements included in the closed circuit. That is, writing $\sum_0$ to denote summation round

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* M_{rs} will have a sign depending on the way in which the coils are wound, and on the convention of sign we set up for the currents in the different circuit elements.

† Usually labour can be saved by specifying the currents, either directly or by the use of mesh currents, so that they satisfy (10.6) automatically.

3

a closed circuit
$$\sum_0 V_r = \sum_0 V,$$

and thus
$$\sum_0 \bar{V}_r = \sum_0 \bar{V}. \qquad . \qquad . \quad (10.8)$$

Using (10.8) in conjunction with (10.3) we have

$$\sum_0 \left\{ z_r I_r + \sum_{s \neq r} z_{rs} I_s \right\} = \sum_0 \bar{V} + \sum_0 \left\{ L_r \dot{I}_r + \sum_{s \neq r} M_{rs} \dot{I}_s - \frac{\dot{Q}_r}{C p} \right\}.$$

$$\qquad \qquad \qquad \qquad \qquad . \qquad \overset{2}{} \qquad . \quad (10.9)$$

By choosing suitable closed circuits, sufficient equations of types (10.9) and (10.7) can be found to give all the I_r.

When voltage V, applied to the terminals of any network with zero initial conditions, produces at the terminals current I given by

$$z(p)I = \bar{V}, \qquad . \qquad . \qquad . \quad (10.10)$$

we call z the "generalized impedance" of the network viewed from these terminals ; [*] for example, z_r defined in (10.4) is the generalized impedance of an L_r, R_r, C_r circuit. Now for a single circuit element of a network with no mutual inductance and with zero initial conditions, (10.3) becomes

$$z_r I_r = \bar{V}_r. \qquad . \qquad . \qquad . \quad (10.11)$$

That is, I_r, $\bar{V}_r$, and z_r are connected in the same way as current, voltage, and resistance in direct current theory. Thus in this case the generalized impedance of the network can be written down by combining the z_r of the circuit elements which comprise it according to the ordinary laws for combining resistances in direct current theory. For example, the generalized impedance of a combination of n generalized impedances $z_1, z_2, \ldots, z_n$ in parallel is given by

$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}. \qquad . \quad (10.12)$$

[*] $z(i\omega)$ is the ordinary complex impedance of alternating current theory.

Again, the generalized impedance of the ladder network, Fig. 5, can, by repeated application of the result for two resistances in parallel, be written down as the continued fraction

$$z = z_1 + \cfrac{1}{\cfrac{1}{z_1'} + \cfrac{1}{z_2 + \cfrac{1}{\cfrac{1}{z_2'} + \cfrac{1}{z_3}}}}. \qquad (10.13)$$
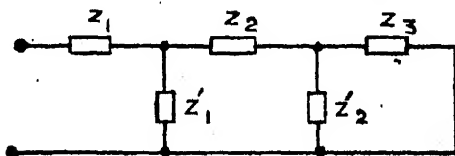
FIG. 5.

It follows in the same way from the remarks at the end of § 8, that the current in any branch of a network, due to initial current $\mathring{I}$ in an inductance and initial charge $\mathring{Q}$ on a condenser in that branch, is given by

$$z(p)\bar{I} = L\mathring{I} - (\mathring{Q}/Cp) \qquad . \qquad (10.14)$$

where $z(p)$ is the generalized impedance of the network viewed from a pair of terminals in the branch.

11. *Examples on electrical networks*

EXAMPLE 1. *The two circuits of Fig. 6 are coupled by mutual inductance M. Voltage V(t) is applied to the primary at t = 0 with zero initial conditions. It is required to find the secondary current (the primary resistance is neglected).*

Here the subsidiary equations (10.3) give

$$L_1 p\bar{I}_1 + Mp\bar{I}_2 = \bar{V},$$

$$Mp\bar{I}_1 + \left(L_2 p + R_2 + \frac{1}{C_2 p}\right)\bar{I}_2 = 0.$$

Thus $I_2 = -\dfrac{Mp\bar{V}}{(L_1L_2 - M^2)p^2 + R_2L_1p + (L_1/C_2)}.$ (11.1)
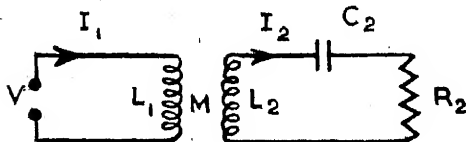
Fig. 6.

If $V = E$, constant, so that $\bar{V} = E/p$, this becomes

$$I_2 = -\frac{ME}{(L_1L_2 - M^2)p^2 + R_2L_1p + (L_1/C_2)},\quad (11.2)$$

and $I_2 = -\dfrac{ME}{\beta(L_1L_2 - M^2)}e^{-\alpha t}\sin\beta t,$

where $\alpha = \dfrac{R_2L_1}{2(L_1L_2 - M^2)},\ \ \beta^2 = \dfrac{L_1}{C_2(L_1L_2 - M^2)} - \alpha^2.$

The case $L_1L_2 = M^2$ is discussed in § 20.

EXAMPLE 2.   *In the circuit of Fig. 7 the condensers are both charged to voltage E, when, at t = 0, the switch S is closed.*

Fig. 7.

We choose the currents as shown in Fig. 7 to satisfy Kirchhoff's first law. Then the subsidiary equation (10.9) for the closed circuits ABD and BGHD are, respectively,

$$Lp\bar{I} + \frac{1}{Cp}\bar{I}_1 = -\frac{CE}{Cp}$$

$$\left(Lp + \frac{1}{Cp}\right)(\bar{I} - \bar{I}_1) - \frac{1}{Cp}\bar{I}_1 = -\frac{CE}{Cp} + \frac{CE}{Cp} = 0.$$

Solving for I, we find

$$I = -\frac{E}{L} \cdot \frac{p^2 + 2n^2}{p^4 + 3n^2 p^2 + n^4}$$

$$= -\frac{E}{2L\sqrt{5}} \left\{ \frac{1+\sqrt{5}}{p^2 + \frac{1}{2}(3 - \sqrt{5})n^2} - \frac{1 - \sqrt{5}}{p^2 + \frac{1}{2}(3 + \sqrt{5})n^2} \right\},$$

where $n^2 = 1/LC$.

Therefore

$$I = -E\sqrt{\left(\frac{C}{10L}\right)} \left\{ \frac{1+\sqrt{5}}{\sqrt{(3 - \sqrt{5})}} \sin nt[\tfrac{1}{2}(3 - \sqrt{5})]^{\frac{1}{2}} \right.$$
$$\left. - \frac{1 - \sqrt{5}}{\sqrt{(3 + \sqrt{5})}} \sin nt[\tfrac{1}{2}(3 + \sqrt{5})]^{\frac{1}{2}} \right\}.$$

EXAMPLE 3. *In the circuit of Fig. 8 it is required to find the current I_1 in the inductance L caused by prescribed current I supplied at the terminals AB, starting at $t = 0$ with zero initial conditions.*
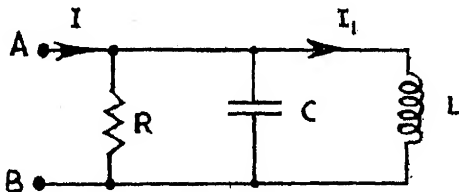
FIG. 8.

From (10.12) it follows that the generalized impedance of the combination of L, R, C in parallel is

$$\frac{1}{Cp + \dfrac{1}{R} + \dfrac{1}{Lp}} = \frac{RLp}{RLCp^2 + Lp + R}.$$

Also the generalized impedance of the inductance L is Lp. Since the same potential is applied to both we hav

$$Lp I_1 = \frac{RLp}{RLCp^2 + Lp + R} I. \qquad \qquad \text{(1}$$

For example if $I = 1$, for $t > 0$, so that $\bar{I} = 1/p$,

$$I_1 = \frac{R}{p(RLCp^2 + Lp + R)},$$

and, if $\beta^2 > 0$,

$$I_1 = 1 - \frac{1}{\beta\sqrt{(LC)}}e^{-\alpha t}\sin(\beta t + \phi),$$

where $\alpha = \dfrac{1}{2RC}$, $\beta^2 = \dfrac{1}{LC} - \alpha^2$, $\phi = \tan^{-1}(\beta/\alpha)$.

EXAMPLE 4. *Bridge networks.*

We consider a general bridge network represented by the scheme of Fig. 9, and suppose known voltage V to be
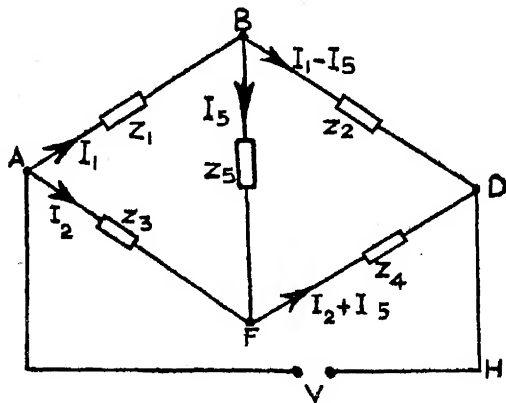
FIG. 9.

applied at $t = 0$ with zero initial conditions. It is required to find I_5, which is the galvanometer current in most applications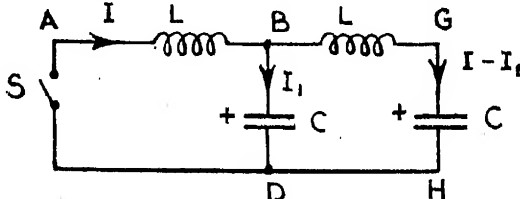. Choosing the currents as shown in Fig. 9 to satisfy Kirchhoff's first law, the subsidiary equations for the closed circuits ABDHA, ABFA, BDFB, respectively, are

$$z_1\bar{I}_1 + z_2(\bar{I}_1 - \bar{I}_5) \qquad\qquad = \bar{V}$$
$$z_1\bar{I}_1 + z_5\bar{I}_5 - z_3\bar{I}_2 \qquad = 0$$
$$z_2(\bar{I}_1 - \bar{I}_5) - z_4(\bar{I}_2 + \bar{I}_5) - z_5\bar{I}_5 = 0.$$

Solving for I_5 we find *

$$I_5 = \frac{\bar{V}}{\Delta} \begin{vmatrix} z_1 + z_2 & 0 & 1 \\ z_1 & -z_3 & 0 \\ z_2 & -z_4 & 0 \end{vmatrix}$$

$$= \frac{\bar{V}}{\Delta}(z_2 z_3 - z_1 z_4), \quad . \quad . \quad . \quad . \quad (11.4)$$

where $\Delta = \begin{vmatrix} z_1 + z_2 & 0 & -z_2 \\ z_1 & -z_3 & z_5 \\ z_2 & -z_4 & -z_2 - z_4 - z_5 \end{vmatrix}$ (11.5)

The way in which the bridge can be used for measurements is determined by the form of $z_2 z_3 - z_1 z_4$ in (11.4) as a function of p. If, by suitable choice of the circuit constants, this can be made to vanish for all p, we have $I_5 = 0$, and thus, by (2.21),

$$I_5 = 0, \ t > 0,$$

whatever the applied voltage V. As an example of this case consider the bridge in which z_1 consists of inductance L and resistance R_1 in series, z_2 of resistance R_2, z_3 of resistance R_3, and z_4 of capacity C and resistance R_4 in parallel. Here

$$z_1 = Lp + R_1, \ z_2 = R_2, \ z_3 = R_3, \ z_4 = \frac{R_4}{1 + R_4 Cp}.$$

Thus $\quad z_2 z_3 - z_1 z_4 = \dfrac{R_2 R_3 (1 + R_4 Cp) - R_4 (Lp + R_1)}{1 + R_4 Cp}$

Then if $\qquad\qquad R_2 R_3 = R_1 R_4,$

and $\qquad\qquad\qquad R_2 R_3 C = L,$

we have $I_5 = 0$ for $t > 0$, whatever the form of the applied voltage may be.

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* It is hardly necessary to point out the advantage of using determinants, particularly when, as here, only one of the currents is needed.

As an example of a bridge which cannot be balanced in this way, consider the case in which

$$z_1 = Lp + R_1, \ z_2 = R_2, \ z_3 = R_3,$$

and z_4 consists of resistance r_1 in series with a parallel combination of resistance r_2 and capacity C, so that

$$z_4 = r_1 + \frac{1}{Cp + (1/r_2)} = \frac{r_1 r_2 Cp + r_1 + r_2}{r_2 Cp + 1},$$

and $z_2 z_3 - z_1 z_4$

$$= -\frac{LCr_1 r_2 p^2 + p\{(R_1 r_1 - R_2 R_3)Cr_2 + L(r_1 + r_2)\} + \{R_1(r_1 + r_2) - R_2 R_3\}}{r_2 Cp + 1} \qquad (11.6)$$

Clearly (11.6) cannot be made to vanish for all values of p by choice of the circuit constants. However, we may use the circuit as an *alternating current bridge* by requiring that the steady state part of I_5 is to vanish when the applied voltage V is sinusoidal and of frequency $\omega/2\pi$. Anticipating the results of § 13, this requires that the real and imaginary parts of the numerator of (11.6), with p replaced by $i\omega$, shall both vanish, that is

$$L(r_1 + r_2) = Cr_2(R_2 R_3 - R_1 r_1) \ \Big\}$$
and
$$\qquad LCr_1 r_2 \omega^2 = R_1(r_1 + r_2) - R_2 R_3 \Big\} \qquad . \quad (11.7)$$

Alternatively the bridge may be used as a *ballistic bridge*; in this case a battery of voltage E is applied at $t = 0$ with zero initial conditions, so that $\bar{V} = E/p$ in (11.4), and the requirement is that the *total* charge passed through the galvanometer is to be zero, that is

$$\int_0^\infty I_5 dt = 0.$$

By Theorem VIII, § 22, this requires

$$\lim_{p \to 0} \bar{I}_5 = 0. \qquad . \qquad . \qquad . \qquad (11.8)$$

It is easy to verify from (11.5) that Δ does not vanish when $p = 0$, so from (11.4) with $\bar{V} = E/p$, and (11.6), it follows that for (11.8) to be satisfied we must have

$$R_2 R_3 = R_1(r_1 + r_2) \Big\}$$
$$R_1 r_2{}^2 C = L(r_1 + r_2) \Big\}.$$

In practice, instead of applying a battery to an initially dead circuit, it is usual to open the battery circuit when steady current is flowing from the battery. In that case the impedances $z_1, \ldots, z_4$ of Fig. 9 must be precisely specified, and the circuit of Fig. 9 (with the arm AHD removed so that $I_2 = -I_1$) has to be studied with known initial values of the currents in the inductances and the charges on the condensers.

12. *The natural frequencies and damping factors of a circuit*

The transforms of the currents, say $\bar{I}_1, \ldots, \bar{I}_n$, in the various branches of an electrical network are found by the solution of equations of types (10.9) and (10.7), and always take the form

$$\bar{I}_r = \frac{f_r(p)}{\Phi(p)}, \ r = 1, \ldots, n, \quad . \quad (12.1)$$

where $\Phi(p)$ is the same for all $\bar{I}_r$, it is in fact the determinant of the coefficients of the $\bar{I}_r$ in the equations and depends on the circuit only. The numerators $f_r(p)$ depend on r, and involve the initial conditions and the applied voltages.

Before we can find the I_r we need the roots of the " period equation "

$$\Phi(p) = 0, \quad . \quad . \quad . \quad (12.2)$$

and we know that to a root $-\mu \pm i\nu$ of (12.2) there corresponds a term $e^{-\mu t} \sin(\nu t + \delta)$ in the I_r, that is, a term of period $2\pi/\nu$ and damping factor $e^{-\mu t}$.

The essential point is that the period equation (12.2), whose roots give the natural frequencies and damping factors of the circuit, is always found in the course of the calculation ·of any of the $\bar{I}_r$, but the $\bar{I}_r$ contain in addition the complete solution of the transient problem. Thus in § 11, Ex. 2, it follows from the first expression for $\bar{I}$ that the natural frequencies of the circuit are $n[\frac{1}{2}(3 \pm \sqrt{5})]^{\frac{1}{2}}/2\pi$.

In demonstration problems, such as most of the examples solved here, $\Phi(p)$ is usually arranged to have obvious linear or quadratic factors. In practice $\Phi(p)$ may be a polynomial in p of degree n with coefficients given numerically,

an example of such a problem is given in § 14. Here we give some references on the methods available for the solution of algebraic equations.

For cubic and quartic equations algebraic methods are available, described in text books on algebra.* For the cubic, some tables and charts are given in Jahnke-Emde.† Numerical methods can be used for cubic and quartic equations, and must be used for equations of higher degree. Real roots can be found from first principles, using a graph and improving the rough values obtained in this way, either by interpolation (an example of this is given in § 14) or by Newton's method (W. & R., § 44). A cubic has always one real root which can be found in this way; dividing by the corresponding factor leaves a quadratic. The more sophisticated methods for finding real roots are Horner's-method (C. S., § 475, W.. and R., § 53) and the root squaring method (W. & R., § 54), the latter is preferable. Problems on resistanceless circuits lead to equations in p^2 with real (negative) roots which are easily solved by the above methods. For equations having several pairs of complex roots, the modified root squaring method seems the most satisfactory; ‡ another method, and also a method of successive approximation to a complex root, is given by Frazer and Duncan.§

If any of the roots of the period equation have positive real parts, the solution will contain terms which increase exponentially with the time; in this case the system is said to be unstable. If all the roots have negative real parts the system is stable. Clearly it will be useful to have a result which will indicate whether a system is stable

* Charles Smith, *Treatise on Algebra*, Edn. 5 (1921), referred to as C. S.; Turnbull, *Theory of Equations* (Oliver and Boyd, 1939). Whittaker and Robinson, *Calculus of Observations* (Blackie, 1924), referred to as W. & R.

† *Tables of Functions* (Edn. 2, Teubner, 1933).

‡ Brodetsky and Smeal, *Proc. Cambridge Phil. Soc.*, 22 (1924), 83. Note that when there are several pairs of complex roots the method given in W. & R. is unsatisfactory.

§ *Proc. Roy. Soc.*, A 125 (1929), 68.

or unstable, without the labour of solving the period equation. One such criterion is Hurwitz's * which states that the roots of the equation

$$a_0 p^n + a_1 p^{n-1} +, \ldots + a_n = 0$$

all have negative real parts if all the determinants

$$a_1, \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \ldots \begin{vmatrix} a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_0 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{2n-1} & a_{2n-2} & \ldots & \ldots & & a_n \end{vmatrix}$$

(12.3)

are positive, where, in writing down the determinants, a_r is put equal to zero if $r > n$.

13. *Steady state alternating currents*

It was remarked in § 12 that if voltage V is applied to any circuit at $t = 0$, the transform of the current I_r in any branch of the circuit will always take the form

$$I_r = \bar{V} \cdot \frac{f_r(p)}{\Phi(p)} + \frac{g_r(p)}{\Phi(p)}, \qquad . \qquad . \quad (13.1)$$

where $f_r(p)$ and $\Phi(p)$ depend on the circuit only, and $g_r(p)$ involves the initial conditions. Now suppose

$$V = E \sin (\omega t + \phi,)$$

so that

$$\bar{V} = E \frac{\omega \cos \phi + p \sin \phi}{p^2 + \omega^2}, \qquad . \qquad . \quad (13.2)$$

and (13.1) becomes

$$I_r = E \frac{(\omega \cos \phi + p \sin \phi) f_r(p)}{(p^2 + \omega^2) \Phi(p)} + \frac{g_r(p)}{\Phi(p)}. \quad (13.3)$$

If the system is a dissipative one, the roots of $\Phi(p) = 0$ will all have negative real parts, and thus will give rise to transient terms. If only the steady state alternating current

* Frank-v. Mises, *Differentialgleichungen der Physik* (Edn. 2, 1930), Vol. 1, p. 162. This is equivalent to Routh's Rule, *Advanced Rigid Dynamics* (Macmillan, Edn. 4, 1884), Chap. vi.

is required, we need only evaluate the two terms of (2.9) corresponding to the zeros $\pm i\omega$ of $p^2 + \omega^2$. These give

$$E\left\{\frac{(\omega \cos \phi + i\omega \sin \phi)f_r(i\omega)e^{i\omega t}}{2i\omega\Phi(i\omega)} + \text{Conjugate}\right\}$$

$$= \frac{E}{Z_r(\omega)} \sin(\omega t + \phi - \delta_r)$$

where $\qquad \dfrac{\Phi(i\omega)}{f_r(i\omega)} = Z_r(\omega)e^{i\delta_r}.$

Thus, if the transform of the current in any part of a circuit due to applied voltage V and any initial conditions has been found, the steady state alternating current can be deduced immediately. An example has been given in (11.7). As another we find the steady state alternating current in the secondary of Fig. 6 due to the primary voltage E $\sin(\omega t + \phi)$. Here (11.1) and (13.2) give

$$I_2 = -\frac{MpE(\omega \cos \phi + p \sin \phi)}{(p^2 + \omega^2)[(L_1L_2 - M^2)p^2 + R_2L_1p + (L_1/C_2)]}.$$

The steady state alternating part of I_2 is

$$-\left\{\frac{Mi\omega E(\omega \cos \phi + i\omega \sin \phi)e^{i\omega t}}{2i\omega[-\omega^2(L_1L_2-M^2)+(L_1/C_2)+R_2L_1i\omega]} + \text{Conjugate}\right\}$$

$$= -\frac{M\omega E}{Z} \cos(\omega t + \phi - \delta),$$

where $\quad Z = \{R_2^2L_1^2\omega^2 + [(L_1/C_2) - (L_1L_2 - M^2)\omega^2]^2\}^{\frac{1}{2}}$

and $\qquad \delta = \arg\{[(L_1/C_2) - \omega^2(L_1L_2 - M^2)] + iR_2L_1\omega\}.$

If $z(p)$ is the generalized impedance of a dissipative circuit looking in at a pair of terminals, and voltage $Ee^{i\omega t}$ is applied at these terminals with zero initial conditions, the transform of the input current will be

$$\frac{E}{(p - i\omega)z(p)},$$

and the steady state current will be

$$\frac{Ee^{i\omega t}}{z(i\omega)}.$$

14. *Numerical treatment of more complicated circuits*

In the problems previously considered, the denominators of the transforms of the solutions have had obvious linear or quadratic factors, so that explicit expressions for the solutions could be written down immediately. In more complicated circuits, the denominators may contain expressions of the third or higher degree in p which have no simple factors. If the circuit constants are given numerically, the zeros of the denominator can be found by one of the methods mentioned in § 12, and the solution completed; if the circuit constants are unspecified, families of curves can be obtained which will show the behaviour of the circuit under any circumstances.
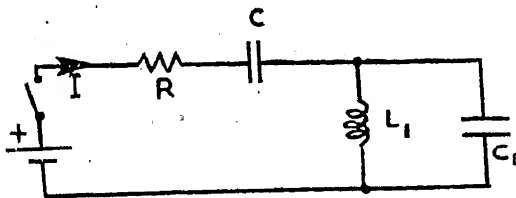
FIG. 10.

In this section we study, from both these points of view, the current I in the circuit of Fig. 10, due to unit voltage applied at $t = 0$ with zero initial conditions. In the usual way we find

$$I = \frac{C(L_1 C_1 p^2 + 1)}{RCL_1 C_1 p^3 + L_1(C + C_1)p^2 + RCp + 1}. \quad (14.1)$$

As an example in which the circuit constants are given, suppose these are: $C_1 = 10^{-9}$, $C = 2.5 \times 10^{-9}$ farads; $L_1 = 0.1$, henrys; $R = 10,000$, ohms. Then (14.1) becomes

$$I = \frac{10^{-4}(p^2 + 10^{10})}{p^3 + 1.4 \times 10^5 p^2 + 10^{10} p + 0.4 \times 10^{15}}. \quad (14.2)$$

To evaluate I, the roots of the period equation

$$p^3 + 1.4 \times 10^5 p^2 + 10^{10} p + 0.4 \times 10^{15} = 0 \quad (14.3)$$

are needed. Putting $p = 10^5 x$ in this, gives the more manageable equation

$$f(x) = x^3 + 1.4x^2 + x + 0.4 = 0. \qquad . \quad (14.4)$$

This cubic can be solved by any of the methods of § 12, but it is very simple to work entirely from first principles. From a graph it is found that there is a real root at approximately $x = -0.77$. This root can be found more precisely as follows: we find $f(-0.77) = 0.00353$, $f(-0.78) = -0.00279$, so that the root lies between -0.77 and -0.78; further, if $f(x)$ ran linearly between its values at these two points it would vanish at -0.776. By repeating this process, as many places of decimals as desired can be found. Working with three places of decimals only, we divide $f(x)$ by $(x + 0.776)$, and obtain

$$f(x) = (x + 0.776)(x^2 + 0.624x + 0.516)$$
$$= (x + 0.776)[(x + 0.312)^2 + (0.647)^2]. \quad (14.5)$$

Returning to (14.2) we find, using Theorem II as in § 2, Ex. 5, that the function whose Laplace transform is

$$\frac{p^2 + a}{(p + \alpha)[(p + \xi)^2 + \eta^2]}$$

is $\quad \dfrac{a + \alpha^2}{(\xi - \alpha)^2 + \eta^2} e^{-\alpha t}$

$$+ \frac{1}{\eta}\left[\frac{(\xi^2 - \eta^2 + a)^2 + 4\xi^2\eta^2}{(\alpha - \xi)^2 + \eta^2}\right]^{\frac{1}{2}} e^{-\xi t} \sin(\eta t - \delta), \quad (14.6)$$

where $\quad \delta = \arg(\xi^2 + a - \eta^2 + 2i\xi\eta) + \arg(\alpha - \xi + i\eta). \quad (14.7)$

Using the values $\alpha = 0.776 \times 10^5$, $\xi = 0.312 \times 10^5$, $\eta = 0.647 \times 10^5$, $a = 10^{10}$, found above, and writing $t = 10^{-6}T$, so that T is the time in micro-seconds, (14.2) and (14.6) give finally

$$10^4 I = 2.527 e^{-0.0776T} + 1.533 e^{-0.0312T} \sin(0.0647T - 1.485)$$
$$(14.8)$$

To plot I from (14.8) we notice that $0.0647T = \pi/8$ when $T = 6.07$, so that steps of 6.07 in T correspond to

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increases of $22\frac{1}{2}°$ in the argument of the sine, and only eight values of the sine need to be looked up. The calculation may be set out as follows, A and B being the first and second terms on the right hand side of (14.8).

T	0·0776T	$e^{-\alpha t}$	A	0·0312T	$e^{-\xi t}$	$\eta t - \delta$	sin	B	$10^4 I$
0	0	1	2·527	0	1	−85·1°	−0·996	−1·527	1·000
6·07	0·471	0·624	1·577	0·189	0·828	−62·6°	−0·888	−1·127	0·450
12·14	0·942	0·390	0·986	0·378	0·685	−40·1°	−0·644	−0·676	0·310
18·21	1·413	0·243	0·614	0·567	0·567	−17·6°	−0·302	−0·263	0·351
24·28	1·884	0·152	0·384	0·756	0·470	4·9°	0·085	0·061	0·445
30·35	2·355	0·095	0·240	0·945	0·389	27·4°	0·460	0·274	0·514
36·42	2·826	0·059	0·149	1·134	0·322	49·8°	0·765	0·378	0·527

It very frequently happens in practice that it is not the transient behaviour of a circuit for given numerical values of the circuit constants which is needed, but a general description of the transient behaviour for any values of the circuit constants. The circuit of Fig. 10 will now be studied from this point of view. The first thing to do is to express (14.1) in a form involving only simple and fundamental quantities, such as dimensionless ratios, time constants of parts of the circuit, etc. Thus, writing

$$n_1{}^2 = \frac{1}{L_1 C_1}, \quad b = \frac{C + C_1}{C_1}, \quad n = \frac{1}{RC}, \quad . \quad (14.9)$$

(14.1) becomes

$$R\bar{I} = \frac{p^2 + n_1{}^2}{p^3 + nbp^2 + n_1{}^2 p + nn_1{}^2}. \quad . \quad (14.10)$$

A further simplification comes from expressing (14.10) in terms of $p_1 = p/n_1$, so that the solution is obtained in terms of $n_1 t$ by Theorem V. Thus writing

$$p_1 = p/n_1, \quad k = n/n_1, \quad . \quad . \quad (14.11)$$

we have

$$n_1 R\bar{I} = \frac{p_1{}^2 + 1}{p_1{}^3 + kbp_1{}^2 + p_1 + k}. \quad . \quad (14.12)$$

The right hand side of (14.12) contains the two, dimensionless, parameters b and k; to study the general behaviour

of the circuit we can regard one of these as fixed at some arbitrary value, and see how the solution changes as the other takes all possible values. We shall give b fixed values, and for each of these allow k to run from o to ∞, that is, say, R to decrease from ∞ to zero.

First we have to study the position of the roots of the period equation

$$x^3 + kbx^2 + x + k = 0 \quad . \qquad . \quad (14.13)$$

for constant b and $k \geqslant 0$. We could give k a number of values and solve (14.13) for each of these, but this would be very laborious, and also we could not be sure that the values of k chosen would fairly represent all types of behaviour in the circuit. Both these difficulties can be avoided by the following procedure: the cubic (14.13) has always one negative real root, call this $-\alpha$, and suppose that it has also a pair of complex roots $-\xi \pm i\eta$ of modulus r. Then we know that * the sum of the roots of (14.13) equals $-kb$, the sum of their products two at a time equals I, and the product of the roots equals $-k$. Or, written in full,

$$\alpha + 2\xi = kb. \qquad . \qquad . \quad (14.14)$$
$$r^2 + 2\alpha\xi = 1 \quad . \qquad . \qquad . \quad (14.15)$$
$$\alpha r^2 = k, . \qquad . \qquad . \quad (14.16)$$
where
$$r^2 = \xi^2 + \eta^2. \qquad . \quad (14.17)$$

If we regard α as a parameter, and solve (14.14) to (14.16) for ξ, r^2 and k in terms of it, we find

$$r^2 = \frac{1 + \alpha^2}{1 + b\alpha^2}, \xi = \frac{\alpha(b - 1)}{2(1 + b\alpha^2)}, k = \frac{\alpha(1 + \alpha^2)}{1 + b\alpha^2}. \quad (14.18)$$

That is, $-\alpha$, $-\xi \pm i\sqrt{(r^2 - \xi^2)}$, with the values of r^2 and ξ given by (14.18), are the roots of (14.13) with the

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* Charles Smith, *Treatise on Algebra*. If α_1, ..., α_n are the roots of

$$x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$$

then $\qquad \Sigma\alpha_1 = -a_1, \qquad \Sigma\alpha_1\alpha_2 = a_2, \qquad \Sigma\alpha_1\alpha_2\alpha_3 = -a_3, \ldots.,$
$$\alpha_1\alpha_2 \ldots \alpha_n = (-)^n a_n.$$

value of k given by (14.18). If the value of ξ given by (14.18) turns out to be greater than r^2, the same argument shows that there are three real roots,

$$-\alpha, \text{ and } -\xi \pm \sqrt{(\xi^2 - r^2)}$$

for the corresponding value of k.
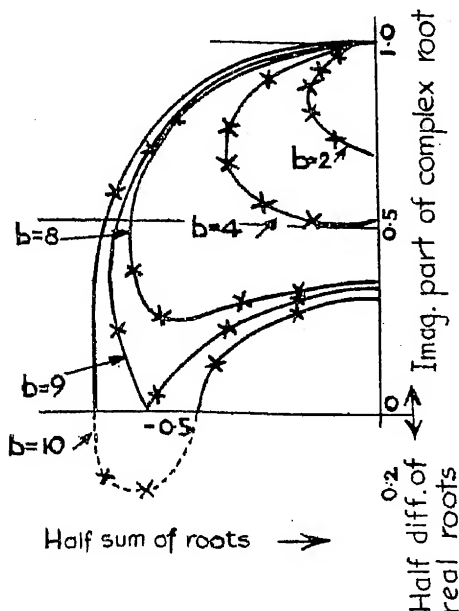
FIG. 11.

In Fig. 11 the position of the complex root $-\xi + i\eta$ of (14.13) is plotted as a function of the parameter α, for the values 2, 4, 8, 9, 10 of b. When $\alpha = 0$ this root is i for all values of b, and as α increases the curves shown are traced out. The positions of the roots for the values 0·2, 0·4, 0·6, 1·0, 2·0 of α are marked by successive crosses on the curves. k is given by (14.18), and, when there is

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a complex root, increases as α increases, that is, as we move round the curves. If $b < 9$ there is always a pair of complex roots of (14.13). If $b = 9$ there is a real triple root for $k = 1/(3\sqrt{3})$ at $\alpha = 1/\sqrt{3}$; for all other values of k there is a pair of complex roots. If $b > 9$ there are three real roots for some values of k; these are indicated by plotting (dotted in Fig. 11) half the sum of the real roots other than α as abscissa, and half their difference as ordinate, measured below the axis.

Thus a diagram such as Fig. 11 gives an immediate indication of the way in which the roots of (14.13) vary
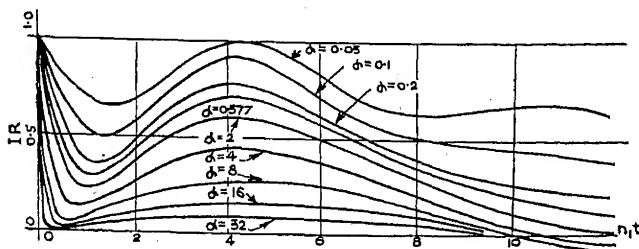
FIG. 12.

with k and b. To study the way in which the current I derived from (14.12) varies with k and b, we select typical values of b from Fig. 11, for example, the transition value $b = 9$, a value of b less than 9, and a value greater than 9. On each of these curves we choose a few representative points and calculate I for these using (14.6).

In Fig. 12 the resulting set of curves for the case $b = 9$ is shown. Here RI is plotted against n_1t for the values 0·05, 0·1, 0·2, $1/\sqrt{3}$, 2, 4, 8, 16, 32 of α, to which correspond the values 0·049, 0·093, 0·152, 0·192, 0·270, 0·469, 0·901, 1·784, 3·558 of k. Clearly these curves, together with two or three such sets for other values of b, will give a fairly complete indication of the way in which the transient behaviour of the circuit depends on k and b.

15. *Circuits which include vacuum tubes*

Transient problems in valve circuits are easily treated by the Laplace transformation method, provided, of course, that the valves are operated in linear parts of their characteristics. Only triodes are considered here, but similar considerations apply to other valves.

The characteristic equation of the triode supplies a linear relation between the anode current, I_a, and the anode, grid, and cathode voltages V_a, V_g, V_c, respectively, namely

$$\rho I_a = (V_a - V_c) + \mu(V_g - V_c), \qquad . \quad (15.1)$$

where ρ is the anode slope resistance, and μ the amplification factor. In transient problems, the quantities V_a, V_g,
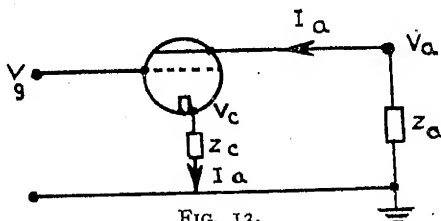
FIG. 13.

V_c, I_a appearing in (15.1) are the variable parts of the voltages and current caused by the transient behaviour being studied; they are superposed on the constant voltages and anode current caused by the anode, and grid bias, batteries. The grid current is negligible in normal circumstances, and here is always assumed to be zero. In any problem it must be decided whether the interelectrode capacities in the valve itself can be neglected—if not, they are easily included as shown below. By virtue of (15.1) it is possible to replace the valve by an equivalent circuit element consisting of resistance ρ and a voltage $-\mu(V_g - V_c)$ inserted between anode and cathode; this is the common practice, but it is a little more fundamental to work directly from (15.1) as will be done here.

As a first example we consider the type of problem that occurs in the amplification of transients: in the circuit of Fig. 13 interelectrode capacities are neglected, and

steady conditions prevail due to grid bias and plate batteries (not shown in the figure), when at $t = 0$ a known additional voltage V_g is applied to the grid ; it is required to find the change, I_a, in the anode current. Since V_g and the changes in currents and voltages it causes are superposed on the steady values, we do not need to know the latter, and need only write down equations for the changes, all of which vanish at $t = 0$. All the currents and voltages marked in Figs. 13-16 are to be understood in this sense. We notice that (15.1) implies a convention of sign, namely, if $V_a > V_0$, I_a is flowing in the direction of the arrow in Fig. 13 ; the same current I_a also flows from the cathode.
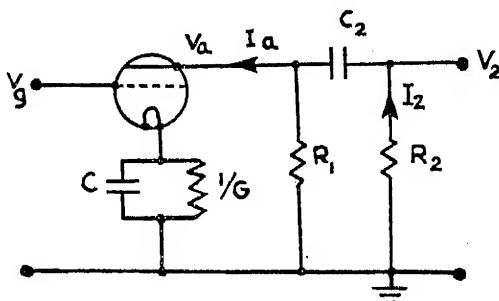
FIG. 14.

The subsidiary equations for the impedances z_a and z_0 are
$$z_a \bar{I}_a = - \bar{V}_a, \qquad . \qquad . \qquad (15.2)$$
$$z_0 \bar{I}_a = \bar{V}_c, \qquad . \qquad . \qquad (15.3)$$
the signs being determined by the convention stated above. Also the transform of (15.1) is
$$\rho \bar{I}_a = \bar{V}_a + \mu \bar{V}_g - (1 + \mu) \bar{V}_0. \qquad . \qquad (15.4)$$
Solving (15.2), (15.3) and (15.4) we find
$$\{\rho + z_a + (1 + \mu) z_0\} \bar{I}_a = \mu \bar{V}_g, \qquad . \qquad (15.5)$$
and from this I_a can be found if V_g is known.

As a more definite problem we take the case of resistance-

capacity coupling, Fig. 14, and find the value of V_2 (which would be the grid voltage of a second valve) corresponding to constant voltage E applied at $t = 0$ to the grid of the first valve. Here we have, in the notation of Fig. 13,

$$z_c = \frac{1}{G + Cp}, \quad z_a = \frac{R_1(1 + R_2C_2p)}{1 + C_2(R_1 + R_2)p}, \quad \bar{V}_g = \frac{E}{p}.$$

Also $\quad \bar{V}_2 = - R_2\bar{I}_2 = - \dfrac{R_1R_2C_2p\bar{I}_a}{1 + C_2(R_1 + R_2)p}$

$$= -\frac{\mu E(G+Cp)R_1R_2C_2}{[1+\mu+p(G+Cp)][1+C_2(R_1+R_2)p]+R_1(1+R_2C_2p)(G+Cp)},$$
$$(15 \cdot 6)$$
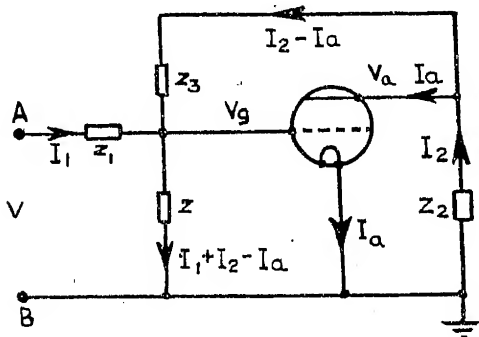
FIG. 15.

where (15.5) has been used in the last line of (15.6). Since the denominator of (15.6) is a quadratic in p, it is easy to write down V_2.

To illustrate the effect of inter-electrode capacity and a rather more complicated system, we consider the circuit of Fig. 15. Here grid-plate capacity is included in z_3, grid-cathode capacity in z, and anode-cathode capacity in z_2. We seek the anode voltage V_a due to voltage V applied at $t = 0$ at the terminals AB when steady conditions prevail.

Choosing currents as shown in Fig. 15 to give zero grid current, the subsidiary equations are

$$z_1 I_1 = \bar{V} - \bar{V}_g \qquad . \qquad (15.7)$$

$$z(I_1 + I_2 - I_a) = \bar{V}_g \qquad . \qquad . \qquad (15.8)$$

$$z_3(I_2 - I_a) = \bar{V}_a - \bar{V}_g \qquad (15.9)$$

$$z_2 I_2 = -\bar{V}_a, \qquad . \qquad . \qquad (15.10)$$

and the transform of (15.1), which is, in this case,

$$\rho I_a = \bar{V}_a + \mu \bar{V}_g. \qquad . \qquad . \qquad (15.11)$$
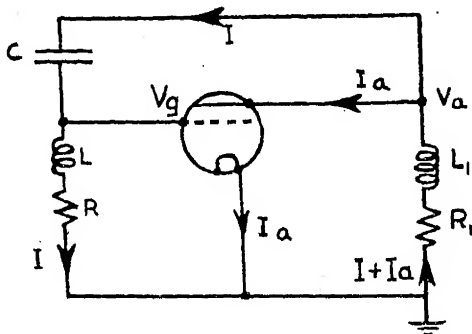
FIG. 16.

Solving we find

$$\bar{V}_a = \frac{\bar{V}zz_2(\rho - \mu z_3)}{zz_1z_2(\mu+1)+(z+z_1)[z_2z_3+\rho(z_2+z_3)]+\rho zz_1}, \qquad (15.12)$$

whence V_a can be found if the impedances z are known.

Finally we treat an oscillator circuit (Fig. 16) from the present point of view. Here there is a closed system in which the grid voltage is not prescribed from outside, but is free to adjust itself. As always, there are steady currents and voltages due to plate and grid bias batteries, and the $I, I_a, V_a, \bar{V}_g$ are departures from these : clearly one solution is to have all of these zero, and the problem is to find how this solution behaves if the system is disturbed. To fix

ideas * suppose there is a change of current $\overset{\bullet}{\mathrm{I}}$ in the inductance L at $t = 0$, then the subsidiary equations are

$$(\mathrm{L}p + \mathrm{R})\bar{\mathrm{I}} = \bar{\mathrm{V}}_g + \mathrm{L}\overset{\bullet}{\mathrm{I}} \qquad . \quad (15.13)$$

$$\frac{1}{\mathrm{C}p}\bar{\mathrm{I}} = \bar{\mathrm{V}}_a - \bar{\mathrm{V}}_g \qquad . \quad (15.14)$$

$$(\mathrm{L}_1 p + \mathrm{R}_1)(\bar{\mathrm{I}}_a + \bar{\mathrm{I}}) = - \bar{\mathrm{V}}_a \qquad . \quad (15.15)$$

$$\rho\bar{\mathrm{I}}_a = \bar{\mathrm{V}}_a + \mu\bar{\mathrm{V}}_g. \qquad . \quad (15.16)$$

The equations (15.13) to (15.16) can be solved for $\bar{\mathrm{I}}, \bar{\mathrm{I}}_a, \bar{\mathrm{V}}_a, \bar{\mathrm{V}}_g$, in terms of $\overset{\bullet}{\mathrm{I}}$ and the circuit constants. All the solutions will have for denominator

$$\Delta(p) = \begin{vmatrix} \mathrm{L}p + \mathrm{R} & 0 & 0 & -1 \\ -1 & 0 & \mathrm{C}p & -\mathrm{C}p \\ \mathrm{L}_1 p + \mathrm{R}_1 & \mathrm{L}_1 p + \mathrm{R}_1 & 1 & 0 \\ 0 & \rho & -1 & -\mu \end{vmatrix}$$

$$= \mathrm{LL}_1\mathrm{C}(1+\mu)p^3 + \mathrm{C}p^2[\rho(\mathrm{L}+\mathrm{L}_1)+(1+\mu)(\mathrm{RL}_1+\mathrm{LR}_1)]$$
$$+ p[\rho\mathrm{C}(\mathrm{R}_1+\mathrm{R})+\mathrm{L}_1+\mathrm{CR}_1\mathrm{R}(1+\mu)]+\rho+\mathrm{R}_1, \quad (15.17)$$

and the solution contains terms in $e^{\alpha_r t}$, where α_r, $r = 1, 2, 3$, are the zeros of (15.17). If all the α_r have negative real parts the disturbance dies out gradually and the system is stable ; if any of the α_r has a positive real part the system is theoretically unstable, actually it executes an oscillation of amplitude determined by the limits to the linearity of the characteristic. The condition for this, which is the condition for the circuit to act as an oscillation generator, is obtained by applying Hurwitz's criterion, (12.3), to (15.17) ; it is

$$\mathrm{C}[\rho(\mathrm{L} + \mathrm{L}_1) + (1 + \mu)(\mathrm{RL}_1 + \mathrm{LR}_1)]$$
$$[\rho\mathrm{C}(\mathrm{R}+\mathrm{R}_1)+\mathrm{L}_1+\mathrm{CRR}_1(1+\mu)] < \mathrm{CLL}_1(1+\mu)(\rho+\mathrm{R}_1).$$

16. *Filter circuits*

The essential feature of such circuits is that a number of precisely similar circuit elements are arranged to form a

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* This is done solely as an illustration, all that is needed is the determinant Δ which does not involve the initial disturbance.

repeating pattern. Mathematically this implies that we can write down a general relation, called a difference equation, connecting the transforms of the currents in successive elements, and by solving this we get a general expression for the current at any point.

As a definite example which includes many cases of practical interest we consider the circuit of Fig. 17. Here there are m impedances z, and $m - 1$ impedances z', arranged as shown. Voltage V is applied at the beginning through impedance z_i, and the whole is terminated by impedance z_0. We require the current I_r at any point due to the voltage V applied at $t = 0$ with zero initial conditions.*
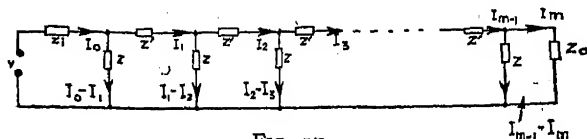
FIG. 17.

The subsidiary equations for the successive meshes of Fig. 17 are

$$(z + z_i)\bar{I}_0 - z\bar{I}_1 = \bar{V} \quad . \quad . \quad . \quad (16.1)$$

$$z\bar{I}_r - (2z + z')\bar{I}_{r+1} + z\bar{I}_{r+2} = 0, \quad r = 0, 1, \ldots, m-2, \quad (16.2)$$

$$-z\bar{I}_{m-1} + (z_0 + z)\bar{I}_m = 0. \quad . \quad . \quad (16.3)$$

The equations (16.2) which connect the transforms of any three consecutive currents are a system of difference equations. We seek a solution of them of type $Ae^{\pm r\theta}$, where A is independent of r. Substituting this in (16.2) gives

$$z - (2z + z')e^{\pm\theta} + ze^{\pm 2\theta} = 0,$$

or

$$z(e^\theta + e^{-\theta}) - (2z + z') = 0,$$

that is

$$\cosh \theta = 1 + \frac{z'}{2z}. \quad . \quad . \quad (16.4)$$

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* If the initial conditions are not zero, as in the discharge of a charged filter circuit, there will be terms on the right hand sides of the subsidiary equations which depend on the particular impedances z and z' in the circuit.

Thus, if θ is defined by (16.4), a solution of (16.2), and in fact the general solution * of it, is

$$I_r = Ae^{r\theta} + Be^{-r\theta}. \qquad . \qquad . \qquad (16.5)$$

It follows from (16.4) that $\sinh^2 \frac{1}{2}\theta = z'/4z$, and, since both signs of θ occur in (16.5), the square root with the positive sign may be taken, which gives

$$\sinh \tfrac{1}{2}\theta = \left(\frac{z'}{4z}\right)^{\frac{1}{2}}. \qquad . \qquad . \qquad (16.6)$$

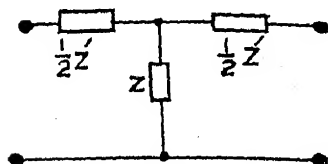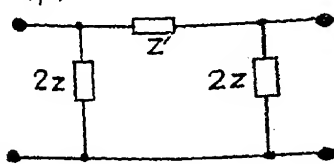FIG. 18 (a).   FIG. 18 (b).

To find the constants A and B in (16.5), this must be substituted in (16.1) and (16.3), which give two equations for A and B, namely

$$(z_i + z - ze^{\theta})A + (z_i + z - ze^{-\theta})B = \bar{V}$$

$$A(z_0 + z - ze^{-\theta})e^{m\theta} + B(z_0 + z - ze^{\theta})e^{-m\theta} = 0.$$

Solving these and substituting in (16.5) gives finally

$$\bar{I}_r = \bar{V}$$
$$\frac{(z+z_0)\sinh(m-r)\theta - z\sinh(m-r-1)\theta}{(z+z_i)(z+z_0)\sinh m\theta - z(z_0+z_i+2z)\sinh(m-1)\theta + z^2\sinh(m-2)\theta}.$$
$$(16.7)$$

This general expression simplifies greatly when z_0 and z_i have values corresponding to the common terminations. The network of Fig. 17 may be regarded as built up of either T elements (Fig. 18a) or $\prod$ elements (Fig. 18b)

* The theory of difference equations is analogous to that of differential equations. (16.5) corresponds to the complementary function, and, if there had been functions of r on the right hand side of (16.2), a particular integral would have had to be included. A brief reference to the theory is given in Piaggio, *Differential Equations*, p. 215. The Laplace transformation method can be applied to difference equations, cf. Gardner and Barnes, *loc. cit.*

connected in tandem, and the general expression (16.7) takes specially simple forms when circuits built up in this way are open or short circuited. For example:

(i) $z_0 = z_i = 0$: Fig. 17 with no series impedance, short circuited at the far end,

$$\mathbf{I}_r = \frac{\bar{\mathbf{V}} \cosh (m - r - \frac{1}{2})\theta}{2z \sinh (m - 1)\theta \sinh \frac{1}{2}\theta}. \qquad . \quad (16.8)$$

(ii) $z_0 = \infty$, $z_i = 0$. The same, but open circuit at the far end,

$$\mathbf{I}_r = \frac{\bar{\mathbf{V}} \sinh (m - r)\theta}{2z \cosh (m - \frac{1}{2})\theta \sinh \frac{1}{2}\theta}. \qquad . \quad (16.9)$$

(iii) $z_0 = z_i = \frac{1}{2}z'$. A circuit of m, T sections, with no additional series impedance and short circuited at the far end. Here, using (16.4) we get

$$\mathbf{I}_r = \frac{\bar{\mathbf{V}} \cosh (m - r)\theta}{z \sinh \theta \sinh m\theta}. \quad . \qquad . \quad (16.10)$$

(iv) $z_i = \frac{1}{2}z'$, $z_0 = \infty$. The same as (iii), but with open circuit at the far end,

$$\mathbf{I}_r = \frac{\bar{\mathbf{V}} \sinh (m - r)\theta}{z \sinh \theta \cosh m\theta}. \quad . \qquad . \quad (16.11)$$

(v) $z_i = z'$, $z_0 = z' + 2z$. A circuit of $m + 1$, Π sections, with no additional series impedance, and open circuited at the far end,

$$\mathbf{I}_r = \frac{\bar{\mathbf{V}} \sinh (m - r + \frac{1}{2})\theta}{2z \cosh (m + 1)\theta \sinh \frac{1}{2}\theta}. \qquad . \quad (16.12)$$

If $\bar{\mathbf{V}}$, z, z', z_0, z_i are known functions of p, (16.4) gives $\cosh \theta$ as a function of p, and using the formulæ

$$\cosh n\theta = 2^{n-1}\left(\cosh \theta - \cos \frac{\pi}{2n}\right)\left(\cosh \theta - \cos \frac{3\pi}{2n}\right)$$
$$\cdots \left(\cosh \theta - \cos \frac{(2n - 1)\pi}{2n}\right) \quad (16.13)$$

$$\sinh n\theta = 2^{n-1} \sinh \theta \left(\cosh \theta - \cos \frac{\pi}{n}\right)\left(\cosh \theta - \cos \frac{2\pi}{n}\right)$$
$$\cdots \left(\cosh \theta - \cos \frac{(n - 1)\pi}{n}\right), \quad (16.14)$$

it appears that (16.7) can always be expressed as a quotient of polynomials in p. Both (16.13) and (16.14) hold for any positive integer n, even or odd. It is often advisable to use them to see the exact form of the denominators of the transforms of the currents, though actual evaluations of the currents I_r are best carried out by working in terms of θ as in the examples below.

EXAMPLE I. Constant voltage E applied to a circuit of m, T sections in which $z' = R$, $z = 1/Cp$, with no additional series impedance, and open circuit at the far end.

Here (16.4) gives

$$\cosh \theta = 1 + \tfrac{1}{2} RCp. \qquad . \qquad (16.15)$$

And, using (16.11),

$$I_r = \frac{CE \sinh (m-r)\theta}{\sinh \theta \cosh m\theta}. \qquad . \qquad (16.16)$$

To evaluate I_r we need to know the zeros of the denominator of (16.16). These can be found from (16.13) with the value (16.15) of $\cosh \theta$, or by noticing that $\cosh m\theta$ is zero when

$$\theta = \frac{(2s+1)i\pi}{2m}, \quad s = 0, \quad 1, \ldots, \quad m-1. \quad (16.17)$$

Taking more values of s than those in (16.17) would result in counting the same zeros over again. By (16.14), $\sinh \theta$ is a common factor of the numerator and denominator of (16.16) so $\theta = 0$ need not be considered in using Theorem II.

The values of p corresponding to the values of θ given by (16.17) are, using (16.15),

$$\alpha_s = -\frac{2}{RC}\left[1 - \cos\frac{(2s+1)\pi}{2m}\right], \; s = 0, \; 1, \ldots, m-1. \quad (16.18)$$

To evaluate I_r it is best to use the form (2.8) of Theorem II, and for this we need

$$\left[\frac{d}{dp}(\sinh\theta\cosh m\theta)\right]_{p=\alpha_s}$$

$$=\left[\frac{d}{d\theta}(\sinh\theta\cosh m\theta)\cdot\frac{d\theta}{dp}\right]_{p=\alpha_s}$$

$$=\left[\tfrac{1}{2}RC\,\frac{\cosh\theta\cosh m\theta+m\sinh\theta\sinh m\theta}{\sinh\theta}\right]_{p=\alpha_s}$$

$$=\tfrac{1}{2}RCm\sinh\frac{(2s+1)i\pi}{2}$$

$$=\tfrac{1}{2}imRC(-)^s.\qquad.\qquad.\qquad.\qquad.\qquad(16.19)$$

Then, applying (2.8) to (16.16), using (16.19), gives

$$I_r=\frac{2E}{mR}\sum_{s=0}^{m-1}(-)^s\,e^{\alpha_s t}\sin\frac{(m-r)(2s+1)\pi}{2m}$$

$$=\frac{2E}{mR}\sum_{s=0}^{m-1}e^{\alpha_s t}\cos\frac{(2s+1)r\pi}{2m}.\qquad.\qquad.\qquad(16.20)$$

EXAMPLE 2. Voltage $V=\sin\omega t$ is applied at $t=0$ to a filter of m, T sections, in which $z'=Lp$, $z=1/Cp$, with open circuit at the far end.

Here (16.4) gives

$$\cosh\theta=1+\tfrac{1}{2}LCp^2.\qquad.\qquad.\qquad(16.21)$$

And from (16.11)

$$I_r=\frac{Cp\omega\sinh(m-r)\theta}{(p^2+\omega^2)\sinh\theta\cosh m\theta}.\qquad.\qquad(16.22)$$

The zeros of the denominator of (16.22) are $\pm i\omega$, and the values of p corresponding to

$$\theta=\frac{(2s+1)i\pi}{2m},\quad s=0,\ 1,\ \ldots,\ m-1.\quad(16.23)$$

Using (16.21) these are seen to be

$$p=\pm i\beta_s,\qquad.\qquad.\qquad.\qquad.\qquad(16.24)$$

where

$$\beta_s=\left(\frac{2}{LC}\right)^{\frac{1}{2}}\left\{1-\cos\frac{(2s+1)\pi}{2m}\right\}^{\frac{1}{2}}\qquad.\qquad(16.25)$$

and it is assumed that none of these equals ω.

Proceeding as in Example 1 we find

$$\left[\frac{d}{dp}\left(\sinh\theta\cosh m\theta\right)\right]_{p=i\beta_s} = (-)^{s+1}\,LCm\beta_s. \quad (16.26)$$

And thus from (2.8) and (16.22)

$$I_r = \frac{2\omega}{mL}\sum_{s=0}^{m-1}\frac{1}{\omega^2-\beta^2{}_s}\cos\beta_s t\,\cos\frac{(2s+1)r\pi}{2m}$$
$$+\left\{\frac{C\omega\sinh(m-r)\theta(i\omega)}{2\sinh\theta(i\omega)\cosh m\theta(i\omega)}e^{i\omega t}+\text{Conjugate}\right\}\quad (16.27)$$

where $\theta(i\omega)$ is the value of θ corresponding to $p=i\omega$.

The second term of (16.27) is the steady state part of period $2\pi/\omega$. To evaluate it, we see from (16.21) that

$$\cosh\theta(i\omega)=1-\tfrac{1}{2}LC\omega^2, \qquad . \quad (16.28)$$

and this gives rise to two possibilities :

(i) if $\qquad\omega^2<4/LC,\ \theta(i\omega)=i\delta$,

where $\qquad\delta=\cos^{-1}(1-\tfrac{1}{2}LC\omega^2)$,

and in this case the part of I_r of period $2\pi/\omega$ is

$$C\omega\frac{\sin(m-r)\,\delta}{\sin\delta\cos m\delta}\cos\omega t. \quad . \qquad . \quad (16.29)$$

(ii) $\qquad$ if $\omega^2>4/LC,\quad\theta(i\omega)=i\pi+\Delta$,

where $\qquad\Delta=\cosh^{-1}(\tfrac{1}{2}LC\omega^2-1)$,

and in this case the part of I_r of period $2\pi/\omega$ is

$$(-)^{r-1}C\omega\frac{\sinh(m-r)\Delta}{\sinh\Delta\cosh m\Delta}\cos\omega t. \quad . \qquad . \quad (16.30)$$

As r is increased, (16.30) decreases exponentially, while on the other hand (16.29) oscillates. That is, if $\omega^2>4/LC$ the term of period $2\pi/\omega$ dies out exponentially as we move along the circuit, while if $\omega^2<4/LC$ this term oscillates trigonometrically. The circuit is, of course, a low pass filter, and we have found both its transient and steady state behaviour for applied voltage $\sin\omega t$.

17. *Vibrations of mechanical systems*

Problems on the motion of bodies of concentrated inertia, connected by shafts or springs of negligible inertia, lead

very readily to systems of ordinary differential equations which have to be solved with given initial conditions. In order to make these differential equations linear, the dissipative forces are restricted to the case of resistance to motion proportional to the velocity, though in some problems Coulomb friction (constant and opposite to the direction of motion) can be treated. Rotating systems only will be considered here, since they are of perhaps a little more interest than the corresponding problems in linear motion, and lead to precisely similar equations.

We write throughout this section, J for the moment of inertia of a wheel about its axis, θ for its angular displacement from a standard position, D for differentiation with respect to time,

$$\omega = D\theta \qquad . \qquad . \qquad . \quad (17.1)$$

for angular velocity, $k\omega$ for resistance to motion proportional to angular velocity, T for torque, and λ for the stiffness of a shaft, that is the torque required to produce unit relative angular displacement between its ends.

The equation of motion of a wheel of moment of inertia J, acted on by torque T, and with resistance to motion $k\omega$, is

$$(JD + k)\omega = T. \qquad . \qquad . \quad (17.2)$$

Also the torque T required to produce a relative displacement θ between the ends of a shaft of stiffness λ is

$$T = \lambda\theta. \qquad . \qquad . \quad (17.3)$$

Using these equations we can write down differential equations for the motion of any combination of wheels connected by shafts of known stiffness. A number of examples of the differential equations which arise in such problems is given below : clearly these can be solved just as in § 5 with given initial values of the ω and θ.

EXAMPLE 1.   A shaft of stiffness λ has one end fixed, and on the other a wheel of moment of inertia J (Fig. 19(a)). Torque T, a given function of time, is applied to the wheel, and there is resistance $k\omega$ to its motion.

If θ is the displacement of the wheel from its equilibrium

position, the retarding torque due to twist of the shaft is $\lambda\theta$, and the differential equation for the motion of the wheel is

$$(JD + k)\omega = T - \lambda\theta. \qquad . \qquad . \quad (17.4)$$
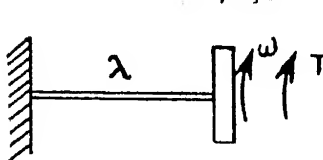
FIG. 19 (a).   FIG. 19 (b).

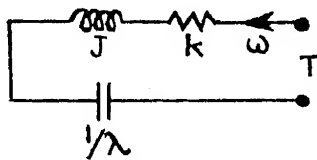Equations (17.1) and (17.4) can be solved with given initial values of θ and ω in the usual way. If preferred, ω can be eliminated ; this gives the second order equation

$$(JD^2 + kD + \lambda)\theta = T. \qquad . \qquad . \quad (17.5)$$
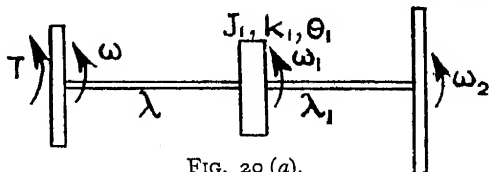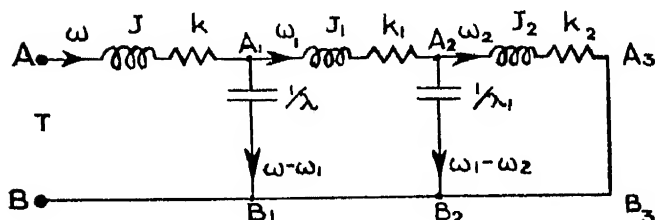

FIG. 20 (a).

FIG. 20 (b).

EXAMPLE 2. Three wheels are connected by shafts of stiffness λ and λ_1 as shown in Fig. 20(a), in which the

symbols used all have the same meanings as before, θ, θ_1, θ_2 being displacements of the wheels from a reference position in which the shafts are unstrained.   Torque T is applied to the wheel J.

Here the differential equations for the motions of the wheels J, J_1, J_2, respectively, are

$$(JD + k)\omega = T + \lambda(\theta_1 - \theta) \quad . \quad . \quad (17.6)$$

$$(J_1D + k_1)\omega_1 = -\lambda(\theta_1 - \theta) + \lambda_1(\theta_2 - \theta_1) \quad (17.7)$$

$$(J_2D + k_2)\omega_2 = -\lambda_1(\theta_2 - \theta_1), \quad . \quad . \quad (17.8)$$

and $\omega = D\theta, \quad \omega_1 = D\theta_1, \quad \omega_2 = D\theta_2. \quad . \quad . \quad (17.9)$

This system can be solved in the usual way with given initial values of θ, θ_1, θ_2, ω, ω_1, ω_2.   Clearly the problem of any number of wheels connected by shafts of given stiffnesses can be solved in the same way ; if all the wheels have the same moment of inertia and the same resistance to motion, and all the shafts the same stiffness, the subsidiary equations form a chain of the type met in § 16 and are treated in the same way.

EXAMPLE 3.   *A geared system.*

First it is necessary to study the type of complication introduced into the system by gearing.   Suppose a gear of radius r_1 drives a gear of radius r_2, their angular velocities being ω_1 and ω_2 in the directions of the arrows in Fig. 21.   Then if θ_1 and θ_2 are the angular displacements of the two gears from their original positions
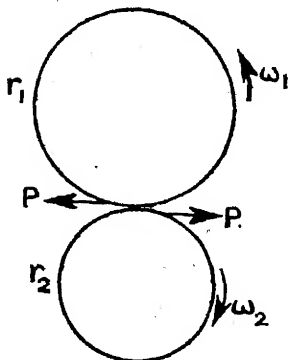
FIG. 21.

$$r_1\theta_1 = r_2\theta_2 \quad . \quad . \quad (17.10)$$

$$r_1\omega_1 = r_2\omega_2. \quad . \quad . \quad (17.11)$$

At the point of contact of the wheels there is a driving force P on the wheel r_2, and

an equal, oppositely directed, reaction force P on the wheel r_1. Then if T_2 is the *driving* torque on the wheel r_2, and T_1 is the *reaction* torque on the wheel r_1

$$T_2 = r_2P, \quad T_1 = r_1P$$

and thus $$r_2T_1 = r_1T_2. \qquad . \qquad . \qquad . \qquad (17.12)$$
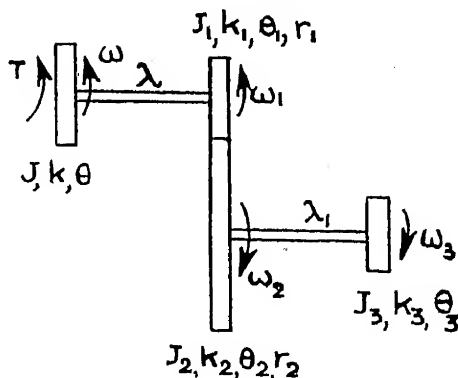
FIG. 22 (a).
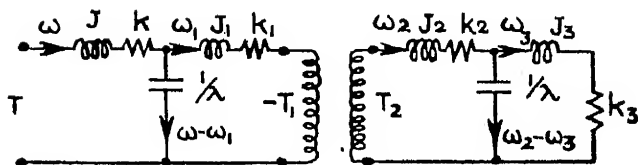
FIG. 22 (b).

(17.11) and (17.12) form a pair of linear connections between the angular velocities and torques in the two shafts. More complicated systems, such as differential gears, give rise to linear connections in the same way.

As an example we consider the system of Fig. 22 (a) in which torque T is applied to the wheel J which drives a gear of radius r_1 and moment of inertia J_1 through a

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shaft of stiffness λ. The gear J_1 meshes with a gear of radius r_2 and moment of inertia J_2, which drives a wheel of moment of inertia J_3 through a shaft of stiffness λ_1. With T_1 and T_2 defined as in (17.12) the equations of motion are

$$(JD + k)\omega = T + \lambda(\theta_1 - \theta) \qquad . \quad (17.13)$$

$$(J_1D + k_1)\omega_1 = -\lambda(\theta_1 - \theta) - T_1 . \quad (17.14)$$

$$(J_2D + k_2)\omega_2 = \lambda_1(\theta_3 - \theta_2) + T_2 \quad . \quad (17.15)$$

$$(J_3D + k_3)\omega_3 = -\lambda_1(\theta_3 - \theta_2). \qquad . \quad (17.16)$$

These equations, together with (17.1), (17.10), (17.11), (17.12) are sufficient to give the motion for any initial conditions consistent * with (17.10) and (17.11).

In many cases it is possible to set up an electric circuit problem which corresponds † to a given mechanical vibration problem, in the sense that both lead to precisely the same systems of differential equations. This does not make the problem any easier to solve, but it gives an interesting physical insight into it, also the corresponding electrical system may be set up and studied in the laboratory, and finally, since the theory of electric circuits is more highly developed than that of mechanical systems, it may be useful in suggesting further developments in that field.

To see the sort of correspondence ‡ which exists between mechanical and circuit problems, compare equations (17.2), (17.3), (17.1), respectively, from which the differential equations for mechanical problems were built up, with the fundamental equations from which the differ-

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* Initial conditions inconsistent with these correspond to forcing into mesh at $t = 0$ a pair of gears inconsistently moving or displaced.

† For a large number of such equivalent problems, cf. Olson, *Dynamical analogies* (van Nostrand, 1943).

‡ The correspondence described here is not the only one; thus, instead of (17.17) we might have used the equation for a leaky condenser, namely $(CD + G)V = I$, and in this way another, equally useful, system arises; cf. Miles, *Journ. Acoust. Soc. America*, **14** (1943), 183.

ential equations of circuit problems are built up, namely :
$$(LD + R)I = V, \qquad . \qquad . \quad (17.17)$$
for an inductive resistance with applied voltage, V ;

$$\frac{1}{C} \cdot Q = V, \qquad . \qquad . \quad (17.18)$$
for a condenser with applied voltage, V ; and
$$DQ = I, \qquad . \qquad . \quad (17.19)$$
for the connection between charge and current. It appears that (17.2), (17.3), (17.1), and (17.17) to (17.19) are the same sets of equations in terms of different quantities, the correspondence between these being set out in the Table below, in which the corresponding quantities for linear motion of a mass are added for completeness.

Electrical.	Rotational.	Linear Motion.
Charge, Q.	Displacement, θ.	Displacement.
Current, I.	Angular velocity, ω.	Linear velocity.
Inductance, L.	Moment of inertia, J.	Mass.
Resistance, R.	Damping coefficient, k.	Damping coefficient.
Capacity, C.	Reciprocal stiffness, $1/\lambda$.	Reciprocal stiffness.
Voltage, V.	Torque, T.	Force.

Using this correspondence we can now set up electric circuits which have the same differential equations as the mechanical examples considered earlier. For example the circuit of Fig. 19 (b) with inductance J, resistance k, and capacity $1/\lambda$, with current ω, charge θ on the condenser, and applied voltage T, gives the same differential equation (17.4) as the mechanical problem of Fig. 19 (a).

Similarly Fig. 20 (b) is the equivalent circuit * to Fig. 20 (a), so that the differential equations found by applying Kirchhoff's second law to the closed circuits AA_1B_1B, $A_1A_2B_2B_1$, and $A_2A_3B_3B_2$, are (17.6) to (17.8) respectively, where θ, θ_1, θ_2 are the amounts of charge which have passed through the inductances J, J_1, J_2 at time t, so that

* Leaky condensers in Fig. 20(b) correspond to resistance to motion proportional to the relative velocities of the wheels in Fig. 20 (a).

$\theta - \theta_1$ and $\theta_1 - \theta_2$ are the charges on the condensers $1/\lambda$ and $1/\lambda_1$ respectively.

Finally we consider the equivalent of the geared system Fig. 22 (a). The linear connections (17.11) and (17.12),

i.e.
$$\frac{\omega_1}{\omega_2} = \frac{T_2}{T_1} = \frac{r_2}{r_1}, \qquad . \qquad . \quad (17.20)$$

become in the electrical case

$$\frac{I_1}{I_2} = \frac{V_2}{V_1} = \frac{r_2}{r_1}, \qquad . \qquad . \quad (17.21)$$

which are the connections between currents and voltages in an *ideal* * transformer of r_2 secondary and r_1 primary turns. On this understanding the equivalent circuit to Fig. 22 (a) is Fig. 22 (b), and it is easy to check that these lead to the same differential equations.

18. *Servomechanisms*

The ideas involved † may be illustrated by considering a rotation control system in which a massive body of moment of inertia J has to be driven so that its displacement θ_0 follows as closely as possible the prescribed displacement $\theta_i(t)$ of a pointer. To do this, voltage $\theta_i - \theta_0$ is fed into an amplifier-motor system which applies torque $k(\theta_i - \theta_0)$ to the body, where k is a constant. This torque tends always to bring the " output displacement " θ_0 into agreement with the " input displacement " θ_i, and is proportional to the " error " $\theta = \theta_0 - \theta_i$. To provide damping for the system, it is usual to " feed-back " into the amplifier a voltage depending on the output velocity $D\theta_0$, or on the error velocity $D\theta$. The simplest arrangement is to mount a generator on the output shaft which provides voltage $k_1 D\theta_0$, which is fed back to the amplifier

* Cf. Starr, *Electric Circuits and Wave Filters* (Pitman, 1934), p. 128.

† Cf. Hazen, " Theory of servo-mechanisms ", *Journ. Franklin Inst.*, **218** (1934), 279 ; Minorsky, " Control Problems ", *ibid.*, **232** (1941), 451, 519.

with negative sign. The torque supplied by the motor is then $k(\theta_i - \theta_0 - k_1 D\theta_0)$, and the equation of motion of the body is

$$JD^2\theta_0 = k(\theta_i - \theta_0 - k_1 D\theta_0). \qquad . \qquad (18.1)$$

If the motion is started at $t = 0$ from rest at zero displacement, the subsidiary equation is

$$(Jp^2 + kk_1p + k)\bar{\theta}_0 = k\bar{\theta}_i, \qquad . \qquad . \qquad (18.2)$$

or

$$\bar{\theta}_0 = \frac{\omega^2\bar{\theta}_i}{p^2 + k_1\omega^2p + \omega^2}, \qquad . \qquad . \qquad (18.3)$$

where

$$\omega^2 = k/J. \qquad . \qquad . \qquad . \qquad . \qquad (18.4)$$

Suppose the input is given a uniform motion $\theta_i = \Omega t$, starting at $t = 0$, so that

$$\bar{\theta}_i = \Omega / p^2, \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (18.5)$$

then (18.3) gives

$$\bar{\theta}_0 = \frac{\omega^2\Omega}{p^2(p^2 + k_1\omega^2p + \omega^2)}$$

$$= \frac{\Omega}{p^2} - \frac{k_1\Omega}{p} + \frac{\Omega(k_1p + k_1^2\omega^2 - 1)}{p^2 + k_1\omega^2p + \omega^2}. \qquad . \qquad (18.6)$$

It follows that

$$\theta_0 = \Omega t - k_1\Omega + \Omega e^{-\frac{1}{2}k_1\omega^2 t}$$
$$\left\{k_1 \cos nt + \frac{1}{n}(\tfrac{1}{2}k_1^2\omega^2 - 1)\sin nt\right\}, \quad (18.7)$$

where

$$n^2 = \omega^2 - \tfrac{1}{4}k_1^2\omega^4.$$

It is seen from (18.7) that the error $\theta = \theta_0 - \theta_i$ in this case is composed of a constant lag, $-k_1\Omega$, together with transient terms.

If instead of feedback $k_1 D\theta_0$, the feedback is arranged to be proportional to the error velocity, $k_1 D\theta$, the equation of motion is

$$JD^2\theta_0 = k(-\theta - k_1 D\theta) \qquad . \qquad . \qquad (18.8)$$

or

$$JD^2\theta + kk_1D\theta + k\theta = -JD^2\theta_i.$$

If the motion is started from rest at zero displacement, the subsidiary equation is

$$(p^2 + k_1\omega^2p + \omega^2)\bar{\theta} = -p^2\bar{\theta}_i, \qquad . \qquad (18.9)$$

where $\omega^2 = k/J$.   If, as before, $\theta_i = \Omega t$, this gives

$$\bar{\theta} = -\frac{\Omega}{p^2 + k_1\omega^2 p + \omega^2},$$

and

$$\theta = -\frac{\Omega}{n}e^{-\frac{1}{2}k_1\omega^2 t}\sin nt. \qquad . \qquad (18.10)$$

In this case the error (18.10) consists of transient terms only and there is no permanent lag as in (18.7).

In practice the voltage $k_1D\theta_0$ or $k_1D\theta$ is usually not fed back directly to the amplifier, but applied to some circuit, and voltage is tapped off from some part of this circuit and fed to the amplifier. In this way rather more complicated transforms arise.

Another complication of practical importance is the presence of time lag in some part of the system. To illustrate the effect of this suppose that, in the first case considered above, the voltage $\theta_i - \theta_0 - k_1D\theta_0$, instead of being fed directly to the amplifier, is applied to an inductance L and resistance R in series, and that the voltage across the resistance is fed to the amplifier as before. Here there is a time delay determined by the time constant of the L, R system. The subsidiary equation for the current I in the L, R circuit, with zero initial conditions is

$$(Lp + R)\bar{I} = \bar{\theta}_i - (1 + k_1p)\bar{\theta}_0, \qquad . \qquad (18.11)$$

and the voltage V across the resistance R is given by

$$\bar{V} = R\bar{I}. \qquad . \qquad . \qquad (18.12)$$

Also the subsidiary equation for the motion of the body of moment of inertia J, with applied torque kV is

$$Jp^2\bar{\theta}_0 = k\bar{V}. \qquad . \qquad . \qquad (18.13)$$

From (18.11) to (18.13) it follows that

$$\bar{\theta}_0 = \frac{\omega^2\omega_1\bar{\theta}_i}{(p^2 + \omega^2)(p + \omega_1) + \omega^2(k_1\omega_1 - 1)p}, \qquad (18.14)$$

where

$$\omega^2 = k/J, \quad \omega_1 = R/L. \qquad . \qquad (18.15)$$

The error θ_0 may be evaluated, for given θ_i, when the roots of the period equation

$$(p^2 + \omega^2)(p + \omega_1) + \omega^2(k_1\omega_1 - 1)p = 0 \qquad (18.16)$$

are known. Hurwitz's criterion (12.3) for the roots of this equation to have negative real parts, that is for the system to be stable, requires

$$k_1 > 1/\omega_1. \qquad . \qquad . \qquad . \quad (18.17)$$

Thus if there is time delay in the system it is necessary for the amount of feedback to be greater than the minimum amount (18.17) or the system will not be stable. The way in which the roots of (18.16) and the values of θ_0 vary as k_1 is increased may conveniently be studied by the method of § 14.

19. *The impulsive function*

In dynamics the idea of a very large force acting for a very short time and such that the time integral of the force is finite is extremely useful, and the " impulsive motion " of dynamical systems is extensively studied. In circuit theory the ideas of an impulsive voltage (that is, a very great voltage applied for a very short time and such that its time integral is finite) or an impulsive current, are equally fruitful.*

It is convenient to have a symbolic representation of such a function and this may be obtained as follows.

Consider the function $\phi(t)$ defined † by

$$\left.\begin{array}{ll} \phi(t) = 0, & t \leqslant 0 \\ \quad = 1/\epsilon, & 0 < t < \epsilon \\ \quad = 0 & t \geqslant \epsilon \end{array}\right\} \qquad . \qquad (19.1)$$

* For an interesting account of impulsive functions and their applications see van der Pol, *Journ. Inst. Elect. Eng.*, **81** (1937), 381.

† Many other functions have the same properties, e.g. the triangular function, Fig. 23 (*b*), or the function

$$(1/\epsilon\sqrt{\pi}) \exp(-t^2/\epsilon^2),$$

Fig. 23 (*c*), except that the latter has its maximum at $t = 0$. It is more usual to choose a function symmetrical about $t = 0$, such as Fig. 23 (*c*), but there is some advantage in the present applications in being able to think of, say, an applied voltage which is zero at $t = 0$ and becomes very large immediately after this instant.

where ϵ is very small. This is very large in a very small region to the right of the origin (Fig. 23 (a)) and zero
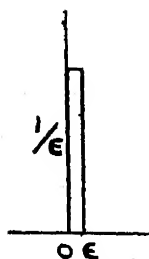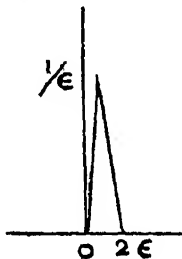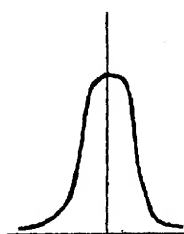
FIG. 23 (a).   FIG. 23 (b).   FIG. 23 (c).

elsewhere, and, no matter how small ϵ is,

$$\int_{-\infty}^{\infty}\phi(t)dt = 1. \qquad . \qquad . \qquad (19.2)$$

Also, if $f(t)$ is a continuous function

$$\int_{-\infty}^{\infty} f(t)\phi(t-a)dt = \frac{1}{\epsilon}\int_{a}^{a+\epsilon} f(t)dt$$
$$= f(a) + O(\epsilon), \qquad . \quad (19.3)$$

where $O(\epsilon)$ is written for a term involving ϵ as a factor, and thus which tends to zero as $\epsilon \to 0$.

We now define the " impulsive " function $\delta(t)$ to possess the properties of $\phi(t)$ as $\epsilon \to 0$ that is :

$\delta(t) = 0$ for $t \leqslant 0$, and for $t > 0$, except for a vanishingly small region to the right of $t = 0$ $\Big\}$, (19.4)

$$\int_{-\infty}^{\infty} \delta(t)dt = 1, \qquad . \qquad . \qquad . \qquad (19.5)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a), \qquad . \qquad . \qquad (19.6)$$

if $f(t)$ is a continuous function.

It follows from (19.6) that

$$\int_0^\infty e^{-pt}\delta(t)dt = 1, \qquad . \qquad . \qquad . \quad (19.7)$$

that is, the Laplace transform of the δ function is unity.

We may now regard a blow of impulse P as a force $P\delta(t)$; an impulsive voltage whose time integral is E, as a voltage $E\delta(t)$; and an impulsive current which transfers charge Q as a current $Q\delta(t)$, and proceed to solve problems involving them in the usual way: the validity of the solutions so obtained is discussed later.

EXAMPLE 1. A mass M attached to a spring of stiffness ω^2M is set in motion at $t = 0$ from rest in its equilibrium position by a blow of impulse P. The equation of motion is

$$MD^2x + M\omega^2x = P\delta(t),$$

and, using (19.7), the subsidiary equation is

$$M(p^2 + \omega^2)\bar{x} = P.$$

Thus $x = (P/M\omega) \sin \omega t.$.   . (19.8)

EXAMPLE 2. An impulsive voltage $E\delta(t)$ is applied at $t = 0$ to an L, R, C circuit with zero initial conditions. Here the subsidiary equation (8.6) is, using (19.7), and the notation (9.3),

$$I = \frac{Ep}{L[(p + \mu)^2 + n^2]}.$$

Thus, in the case $n^2 > 0$,

$$I = \frac{E}{Ln}e^{-\mu t}\{n \cos nt - \mu \sin nt\}. \qquad . \quad (19.9)$$

The solutions of Examples 1 and 2 above have been obtained by the ordinary Laplace transformation procedure, using (19.7) for the transform of $\delta(t)$, and, since any use of the δ function must be regarded as symbolical only, *

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* The δ function was defined as the limit of some function as $\epsilon \to 0$. Any mathematical operations on this function, such as differentiation or integration, would involve the interchange of order of two limiting processes and be very difficult to justify.

we must inquire whether these solutions are mathematically justifiable. · The simplest way to do this is to verify that they do in fact satisfy the differential equations and initial conditions of their problems. Consider the solution (19.8) from this point of view: it satisfies the differential equation

$$MD^2x + M\omega^2x = 0,\ t > 0,\qquad . \quad (19.10)$$

for the motion of the mass M for $t > 0$; also as $t \to 0$ in (19.8), $x \to 0$, and $Dx \to P/M$. Now P/M is the velocity with which, according to the theory of impulsive motion in dynamics, the mass M would be started by a blow of impulse P. Thus the solution obtained by the formal Laplace transformation procedure applied to Example 1 does in fact give the complete solution of the dynamical problem correctly, and in addition it avoids the necessity of evaluating the initial velocity due to the blow. It can easily be verified * that the same is true for a general dynamical ·system of n degrees of freedom set in motion by blows.

Precisely the same is true for the electrical case in Example 2. Here, as $t \to 0$ the value of I given by (19.9) tends to E/L, and we need an electrical analogue of the theory of impulsive motion in dynamics to show that an impulsive voltage $E\delta(t)$, applied to an L, R, C circuit in which no current is flowing, will instantaneously produce a current of this magnitude. This can be deduced from the differential equation for current and charge in the circuit, namely

$$(LD + R)I + \frac{Q}{C} = E\delta(t).\qquad . \quad (19.11)$$

Proceeding as in dynamical theory, we integrate this over a very small time, τ: the integral of $\delta(t)$ over this time is 1; that of the finite quantities I and Q is very small; and that of DI is the difference between the final

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* By an easy extension of the method in Carslaw and Jaeger, *loc. cit.*, § 35. Jaeger, *Phil. Mag.* (7), **36** (1945), 644, gives a complete verification for all the types of problem arising in this section and § 20.

value, I_f, and the initial value (zero) of the current I. Thus

$$LI_f = E, \qquad . \qquad . \qquad . \quad (19.12)$$

and the theory of impulsive motion gives the current $I_f = E/L$, to which the solution obtained by the Laplace transformation procedure was found to tend as $t \to 0$. It is easy to verify that (19.9) satisfies the differential equation of the problem for $t > 0$, so it does in fact give the correct solution of the physical problem.

It can be proved in general that for a complicated circuit with various impulsive applied voltages, the solutions obtained by the Laplace transformation procedure, treating $\delta(t)$ as having (19.7) for transform, do satisfy the equations of circuit theory, and, as $t \to 0$, the currents tend to the values which would be found from the equations of impulsive motion.

20. *Justification of the solutions of electrical network problems*

It was remarked in § 7 that, for a system of linear differential equations, a general verification could be given that the solutions obtained by the Laplace transformation method satisfied the differential equations and initial conditions, provided a certain determinant did not vanish. The problems of circuit theory often give rise to a system of differential equations to which this verification applies immediately. In general, however, they give a combination of differential or algebraic equations (10.1) and algebraic equations (10.6); it may happen that the specified initial values of the currents and charges do not satisfy some of these equations, and in that case there will be an impulsive readjustment of these initial values. It is in fact true * in all cases that the solutions obtained by the Laplace transformation method, used as in § 10, giving each inductance its own initial current, and each condenser its own initial charge, in the subsidiary equations, do in fact give solutions of the equations of the problems

* Jaeger, *loc. cit.*

for $t > 0$, which as $t \to 0$ tend to the values which exist in the circuit after the impulsive readjustment of initial conditions, if this takes place. The general theory will not be given here, but only examples illustrating the different cases which can arise.

EXAMPLE 1. Steady current E/R is flowing in the circuit of Fig. 24 with the switch S closed. At $t = 0$ the switch is opened.
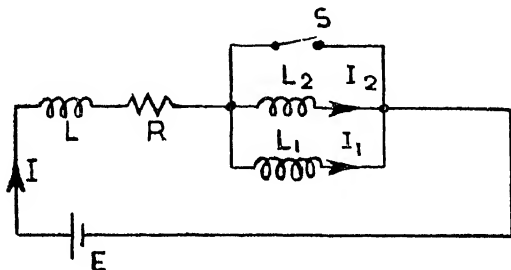
FIG. 24.

Since the initial current is E/R in L, and zero in L_1 and L_2, the subsidiary equations are

$$(Lp + R)\bar{I} + L_2 p \bar{I}_2 = \frac{LE}{R} + \frac{E}{p} \left. \right\}$$
$$L_2 p \bar{I}_2 - L_1 p \bar{I}_1 = 0 \qquad . \quad (20.1)$$
$$\bar{I} = \bar{I}_1 + \bar{I}_2.$$

Solving we find

$$\bar{I} = \frac{E}{Rp} - \frac{L_1 L_2 E}{R\{(LL_1 + LL_2 + L_1 L_2)p + R(L_1 + L_2)\}} \quad (20.2)$$

And thus

$$I = \frac{E}{R} - \frac{L_1 L_2 E}{R(LL_1 + LL_2 + L_1 L_2)} e^{-R(L_1 + L_2)t/(LL_1 + L_1 L_2 + LL_2)} \quad (20.3)$$

As $\qquad t \to 0, \quad I \to \frac{E(LL_1 + LL_2)}{R(LL_1 + LL_2 + L_1 L_2)}, \quad (20.4)$

and, since this does not equal E/R, it appears that there has been an impulsive readjustment of current between the three inductances just after the switch was opened.

To investigate what happens in this case, and to verify this result, consider the equations of the system for $t > 0$, namely

$$(LD + R)I + L_2DI_2 = E, \qquad . \qquad (20.5)$$

$$L_2DI_2 - L_1DI_1 = 0, \qquad . \qquad (20.6)$$

$$I_2 + I_1 = I. \qquad . \qquad (20.7)$$

The values E/R, 0, 0, which the currents I, I_1, I_2 had before the switch S was opened, do not satisfy (20.7). Suppose that these change to $I^{(0)}$, $I_1^{(0)}$ and $I_2^{(0)}$ in a very short time τ following the opening of S at $t = 0$, then, integrating (20.5) and (20.6) from $t = 0$ to $t = \tau$, and neglecting the integrals of the finite quantities E, I, I_1, I_2 over this very small time, we find

$$L\{I^{(0)} - (E/R)\} + L_2I_2^{(0)} = 0, \qquad . \qquad (20.8)$$

$$L_2I_2^{(0)} - L_1I_1^{(0)} = 0. \qquad . \qquad (20.9)$$

Also $I^{(0)}$, $I_1^{(0)}$ and $I_2^{(0)}$ satisfy (20.7), that is

$$I^{(0)} = I_1^{(0)} + I_2^{(0)}. \qquad . \qquad (20.10)$$

Solving (20.8) to (20.10) gives the new values of the currents, and, in particular, $I^{(0)}$ is found to have the value (20.4). If the voltage drops across the inductances were evaluated in the usual way, they would be found to contain δ functions, that is, the finite changes of current may be regarded as being produced by impulsive voltages.

EXAMPLE 2. The condenser C in the circuit of Fig. 25 (p. 78) is charged to potential E, and the condenser C_1 is un-charged, when at $t = 0$ the switch S is closed.

Here the subsidiary equations are

$$\left. \begin{array}{l} \dfrac{1}{Cp}\bar{I} + \dfrac{1}{C_1p}\bar{I}_1 = -\dfrac{E}{p} \\[2mm] \dfrac{1}{C_1p}\bar{I}_1 - R(\bar{I} - \bar{I}_1) = 0. \end{array} \right\} \qquad . \qquad . \qquad (20.11)$$

Solving we find

$$I = -\frac{EC(RC_1 p + 1)}{R(C + C_1)p + 1}$$

$$= -\frac{ECC_1}{C + C_1} - \frac{EC^2}{R(C + C_1)^2} \cdot \frac{1}{p + [1/R(C + C_1)]}.$$

Thus $I = -\dfrac{ECC_1}{C + C_1}\delta(t) - \dfrac{EC^2}{R(C + C_1)^2}e^{-t/R(C+C_1)}.$ (20.12
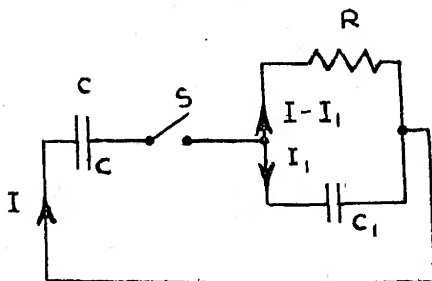
FIG. 25.

The term $[-ECC_1/(C + C_1)]\delta(t)$ in (20.12) represents an impulsive current which instantaneously exchanges charge $ECC_1/(C + C_1)$ between the two condensers; the second term represents the subsequent discharge of the condensers through the resistance. Whenever a network includes a closed circuit containing condensers only, there will be an instantaneous exchange of charge in this way between the condensers if these are incompatibly charged. If we had used the subsidiary equations for charge instead of those for current, this finite change of charge would have been found (in the same way as the finite change of current in Example 1) but as we have worked with current, the impulsive current which transfers this amount of charge has appeared. The result (20.12) can be verified by writing down the differential equations for the charges on the condensers and proceeding as in Example 1.

EXAMPLE 3. Steady current E/R_1 is flowing in the primary of a transformer, Fig. 26, and the secondary
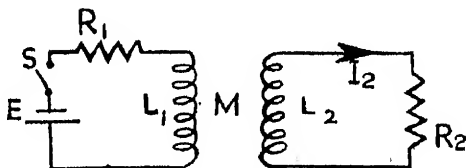
FIG. 26.

current is zero, when at $t = 0$ the switch S is opened. Writing I_1 and I_2 for the primary and secondary currents, the subsidiary equation for the secondary is

$$Mp\bar{I}_1 + (L_2p + R_2)\bar{I}_2 = ME/R_1 \quad . \quad (20.13)$$

Also $\qquad I_1 = 0$, for $t > 0$, so $\bar{I}_1 = 0$.

Thus from (20.13) $\quad \bar{I}_2 = \dfrac{ME}{R_1(L_2p + R_2)}$,

$$\bar{I}_2 = \frac{ME}{R_1 L_2} e^{-R_2 t/L_2}. \quad . \quad . \quad (20.14)$$

Here as $t \to 0$, $I_2 \to ME/R_1L_2$. There was a sudden change in the currents when the switch was opened which could have been studied as in Example 1.

EXAMPLE 4. The problem of § 11, Example 1, for the case of $L_1L_2 = M^2$ (perfect coupling). Here (11.2) becomes

$$\bar{I}_2 = -\frac{ME}{R_2L_1p + (L_1/C_2)}, \quad . \quad (20.15)$$

and thus $\qquad I_2 = -\dfrac{ME}{R_2L_1} e^{-t/R_2C_2} \quad . \quad . \quad (20.16)$

It is known from general theory that $L_1L_2 \geqslant M^2$; the limiting case $L_1L_2 = M^2$ is of interest for two reasons: firstly it may be very nearly approached in practice; secondly, if $L_1L_2 = M^2$, the degree of the denominator in $\bar{I}_2$ is reduced by one, here a quadratic has become linear and similarly a cubic becomes a quadratic, and the algebra is greatly simplified.

The significance of perfect coupling from the theoretical point of view * is best seen by considering the differential equations of the circuit of Fig. 26 : these are

$$\left.\begin{array}{l}(L_1 D + R_1)I_1 + MDI_2 = E\\ MDI_1 + (L_2 D + R_2)I_2 = 0\end{array}\right\}\quad.\quad(20.17)$$

If $L_1 L_2 = M^2$, multiplying the first of (20.17) by L_2, the second by M, and subtracting, we obtain

$$L_2 R_1 I_1 - MR_2 I_2 = L_2 E.\qquad.\quad(20.18)$$

Thus in this case we can form an algebraic equation by elimination from the given differential equations, and if the given initial currents do not satisfy (20.18) they will change suddenly to new values which can be calculated as in Example 1.

The examples above illustrate the types of problem in which there may be sudden changes of charges or currents on closing switches, and the way in which these could have been found has been sketched briefly. A general discussion along these lines can be given, which shows that the Laplace transformation procedure, applied in the usual way, does in fact always give the correct solution of the physical problem.

EXAMPLES ON CHAPTER II

1. Alternating voltage E sin $(nt + \epsilon)$ is applied at $t = 0$ to inductance L and capacity C in series. $n^2 = 1/LC$. If the initial current and charge are zero, show, using (1.11) and (1.12), that the current at any time is

$$(E/2nL)\{nt \sin(nt + \epsilon) + \sin \epsilon \sin nt\}.$$

2. Constant current (E/R) is supplied for $t > 0$ to a parallel combination of inductive resistance L, R and capacity C with zero initial conditions. Show that the voltage drop across the condenser is

$$E - Ee^{-\mu t}\{\cos nt + [(R/2nL) - (1/nRC)]\sin nt\},$$

* This is an example of the exceptional case referred to in § 7, since the determinant discussed there is $L_1 L_2 - M^2$ for the system of equations (20.17).

where μ and n are defined in (9.3). Deduce the solution of § 9, Ex. 3.

3. In the problem of § 11, Ex. 2, show that the charge on the first condenser is

$$(CE/2\sqrt{5})\{(1 + \sqrt{5}) \cos nt[\tfrac{1}{4}(3 - \sqrt{5})]^{\tfrac{1}{2}} \\ - (1 - \sqrt{5}) \cos nt[\tfrac{1}{4}(3 + \sqrt{5})]^{\tfrac{1}{2}}\}.$$

4. Regarding an imperfect condenser as capacity C in series with resistance R_1, the whole being shunted by conductance G, show that the subsidiary equation for voltage V applied to such a condenser in series with an inductive resistance L, R is

$$\left\{Lp + R + \frac{1 + R_1Cp}{Cp + G(1 + R_1Cp)}\right\}\bar{I} = \overline{V} + L\overset{\circ}{I} - \frac{\overset{\circ}{Q}}{Cp + G(1 + R_1Cp)}.$$

5. In the bridge of Fig. 9, z_1 consists of resistance R_1 and capacity C_1 in parallel, z_3 consists of resistance R_3 and capacity C_3 in parallel, and z_2 and z_4 consist of resistances R_2 and R_4 respectively. Show that there is a balance for all types of applied voltage if
$$C_1/C_3 = R_4/R_2 = R_3/R_1.$$

6. In the circuit of Fig. 6, $L_1L_2 = M^2$, and steady-state conditions prevail due to alternating voltage of peak value E and frequency $\omega/2\pi$ in the primary. At the instant of zero secondary current the condenser is instantaneously discharged, show that at time t after this instant the secondary current is

$$\frac{EC_2\sqrt{L_2}}{\sqrt{\{L_1(1 + \omega^2R_2{}^2C_2{}^2)\}}}\left\{\omega \sin \omega t - \frac{1}{R_2C_2}e^{-t/R_2C_2}\right\}.$$

7. In the resistance-capacity coupled amplifier of Figs. 13, 14, with $z_0 = 0$, show that if the grid voltage is changed by E at $t = 0$, the change in grid voltage of the next valve is

$$V_2 = - \frac{\mu R_1R_2E}{R_1R_2 + \rho(R_1 + R_2)}e^{-(\rho + R_1)t/C_2[R_1R_2 + \rho(R_1 + R_2)]}$$

8. A mass M hangs by a spring of stiffness Mn^2. At $t = 0$, when the mass is at rest in its equilibrium position, the other end of the spring is given a downwards motion $a \sin \omega t$. Show that the displacement of the mass from its equilibrium position at time t is

$$\frac{an}{\omega^2 - n^2}[\omega \sin nt - n \sin \omega t].$$

9. Constant voltage E is applied at $t = 0$ to a condenser of capacity C_1 in series with a leaky condenser of capacity C and leakage conductance G. Show that, if the initial charges are zero, the current is

$$\frac{ECC_1}{C + C_1}\delta(t) + \frac{EGC_1{}^2}{(C + C_1)^2}e^{-Gt/(C+C_1)}.$$

6

FURTHER THEOREMS AND THEIR APPLICATIONS

21. *Heaviside's shifting theorem*

THEOREM VI. *If $\bar{y}(p)$ is the transform of $y(t)$, and $a > 0$, then $e^{-ap}\bar{y}(p)$ is the transform of*

$$y(t - a)\mathrm{H}(t - a), \qquad . \qquad . \quad (21.1)$$

where *
$$\begin{aligned} \mathrm{H}(t) &= 0, \quad t < 0 \\ &= 1, \quad t > 0 \end{aligned} \Bigg\} \qquad . \qquad . \quad (21.2)$$
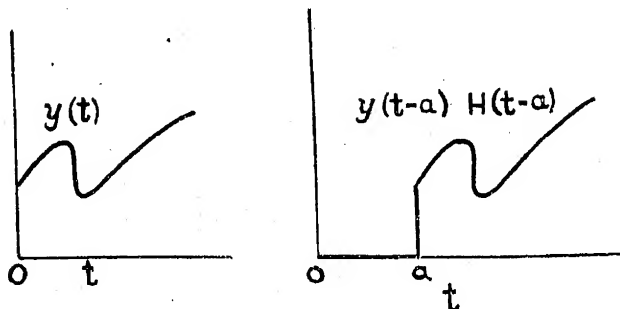
FIG. 27 (*a*).

Since $\mathrm{H}(t - a)$ is zero if $t < a$ and unity if $t > a$, the function $y(t - a)\mathrm{H}(t - a)$ is zero for $t < a$, and for $t > a$ its graph is exactly that of $y(t)$ for $t > 0$, shifted to the right through a distance a, as in Fig. 27 (*a*): hence the name " shifting theorem ".

* This is Heaviside's unit function; the notation (21.2) will be found very useful in many problems. It has not appeared earlier in the present treatment, since problems have been stated hitherto in the form of, say, " unit voltage applied to a circuit at $t = 0$ ", instead of " voltage $\mathrm{H}(t)$ applied to a circuit ": it must be noticed that these statements are identical only for $t > 0$; for $t < 0$ the first does not specify the voltage but the second makes it zero.

To prove (21.1) we have only to notice that

$$\int_0^\infty e^{-pt} y(t-a) H(t-a) dt = \int_a^\infty e^{-pt} y(t-a) dt = \int_0^\infty e^{-ap} e^{-pt} y(t) dt$$
$$= e^{-ap} \bar{y}(p).$$

Using this theorem we can build up the transforms of a large number of useful step and broken functions.
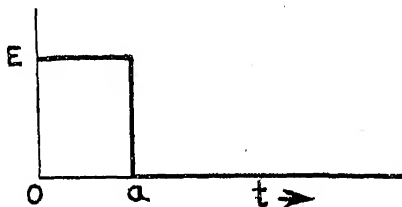
FIG. 27 (b).

EXAMPLE 1. $y(t) = E, \quad 0 < t < a$,
$\qquad\qquad\quad = 0, \quad t > a$

i.e. $y(t) = E\{1 - H(t - a)\}$, cf. Fig. 27 (b),

$$\bar{y} = \frac{E}{p}(1 - e^{-ap}). \qquad . \qquad . \qquad (21.3)$$
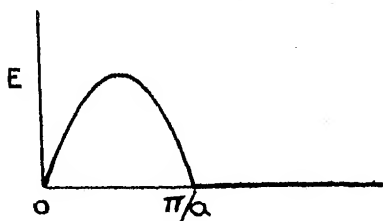
FIG. 27 (c).

EXAMPLE 2.   A half sine wave, Fig. 27 (c),
$\qquad y = E \sin \omega t, \quad 0 < t < \pi/\omega$,
$\qquad\quad = 0, \qquad\qquad t > \pi/\omega$

i.e. $y = E \sin \omega t + E \sin \omega(t - \pi/\omega) H(t - \pi/\omega)$,

$$\bar{y} = \frac{E\omega}{p^2 + \omega^2}(1 + e^{-p\pi/\omega}) \qquad . \qquad . \qquad . \qquad (21.4)$$

EXAMPLE 3.   A " square " wave, Fig. 27

$$y = E, \qquad rT < t < (r + 1)T \biggr\}$$
$$\ \ \ = -E, \ (r+1)T < t < (r+2)T \biggr\}$$

i.e. $y = E\{H(t) - 2H(t - T) + 2H(t - 2$
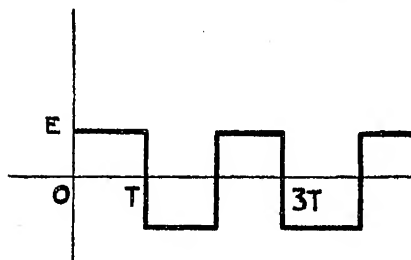
FIG. 27 (d).

$$\bar{y} = \frac{E}{p}\{1 - 2e^{-pT} + 2e^{-2pT} - \ldots\}$$

$$= \frac{E}{p} \cdot \frac{1 - e^{-pT}}{1 + e^{-pT}}$$

$$= \frac{E}{p} \tanh \tfrac{1}{2}pT. \quad . \quad . \quad .$$
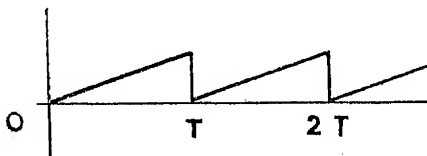
FIG. 27 (e).

EXAMPLE 4.   A " saw-tooth " wave, Fig.

$$y = k(t - rT), \quad rT < t < (r+1)T, \quad$$

$$\bar{y} = \frac{k}{p^2} - \frac{kT}{p}(e^{-pT} + e^{-2pT} + \ldots)$$

$$= \frac{k}{p^2} - \frac{kTe^{-pT}}{p(1 - e^{-pT})}.$$

EXAMPLE 5. A function $f(t)$ of period T, so that $f(t + r\text{T}) = f(t)$, if r is any integer.

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)dt = \int_0^\text{T} e^{-pt} f(t)dt + \int_\text{T}^{2\text{T}} e^{-pt} f(t)dt + \ldots$$

$$= \int_0^\text{T} e^{-pt} f(t)dt + e^{-p\text{T}} \int_0^\text{T} e^{-pt} f(t)dt + \ldots$$

$$= (1 + e^{-p\text{T}} + e^{-2p\text{T}} + \ldots) \int_0^\text{T} e^{-pt} f(t)dt$$

$$= \frac{1}{1 - e^{-p\text{T}}} \cdot \int_0^\text{T} e^{-pt} f(t)dt. \qquad . \quad (21.7)$$

(21·5) and (21.6) can of course be deduced from this result.

EXAMPLE 6. The transforms of the functions in Examples 3 to 5 have all involved hyperbolic functions. A technique which is of great importance both here and in connection with partial differential equations, is that for finding the function which has a given transform of this type. To do this we expand the transform by the Binomial Theorem in a series of negative exponentials : for example if

$$\bar{y}(p) = \frac{\text{E}}{p^2} \tanh p\text{T} = \frac{\text{E}}{p^2} \cdot \frac{1 - e^{-2p\text{T}}}{1 + e^{-2p\text{T}}}$$

$$= \frac{\text{E}}{p^2} (1 - e^{-2p\text{T}})(1 - e^{-2p\text{T}} + e^{-4p\text{T}} - \ldots)$$

$$= \frac{\text{E}}{p^2} (1 - 2e^{-2p\text{T}} + 2e^{-4p\text{T}} - \ldots).$$

Then, by Theorem VI,

$$y(t) = \text{E}t - 2\text{E}(t - 2\text{T})\text{H}(t - 2\text{T})$$
$$+ 2\text{E}(t - 4\text{T})\text{H}(t - 4\text{T}) - \ldots, \quad (21.8)$$

the function y is shown in Fig. 27 (f) (p. 86).

EXAMPLE 7. A battery of voltage E is applied at $t = 0$ to an L, R, C circuit with zero initial conditions, and at $t = \text{T}$ the battery is short-circuited.

Here the voltage in the circuit is $E - EH(t - T)$, and by Example 1 its transform is

$$\frac{E}{p}(1 - e^{-pT}).$$

Then, as in (9.2), the transform of the current is

$$I = \frac{E(1 - e^{-pT})}{L[(p + \mu)^2 + n^2]},$$

and, by Theorem VI,

$$I = \frac{E}{nL}\{e^{-\mu t} \sin nt - e^{-\mu(t-T)}H(t - T) \sin n(t - T)\}, \quad (21.9)$$

for the case $n^2 > 0$.
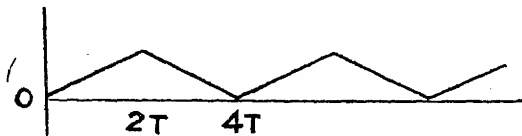
FIG. 27 (f).

The solution for applied voltages such as those of Examples 3 and 4 is found in the same way (cf. Example 5 at the end of this Chapter). An alternative form for the solution in these cases can be obtained by using the expansion formula of § 33.

EXAMPLE 8. To find the deflection y at any point of a light uniform beam of length l, which is clamped horizontally at the same level at its ends, $x = 0$ and $x = l$, and carries a concentrated load W at $x = a$.

Using the δ-function defined in § 19, a concentrated load W at $x = a$ may be treated as a distributed load $W\delta(x - a)$. Then, proceeding as in § 6, we have to solve the differential equation

$$EID^4y = W\delta(x - a), \quad . \quad . \quad (21.10)$$

with $y = Dy = 0$, when $x = 0$ and $x = l$.

The subsidiary equation is, by (19.7) and Theorem VI,

$$\text{EI}p^4\bar{y} = \text{W}e^{-ap} + py_2 + y_3 \qquad . \quad (21.11)$$

and thus, from (21.11) and Theorem VI,

$$y = \frac{\text{W}(x-a)^3}{6\text{EI}}\text{H}(x-a) + \frac{x^2 y_2}{2\text{EI}} + \frac{x^3 y_3}{6\text{EI}},$$

where y_2 and y_3 are unknown constants to be found from the conditions $y = \text{D}y = 0$, at $x = l$. Solving for these we get finally,

$$y = \frac{\text{W}(x-a)^3}{6\text{EI}}\text{H}(x-a)$$
$$+ \frac{\text{W}ax^2(l-a)^2}{2\text{EI}l^2} - \frac{\text{W}x^3(l-a)^2(l+2a)}{6l^3\text{EI}}.$$

22. *Heaviside's series expansion*

This is a useful method for determining the behaviour of the solution for small values of the time—the solutions obtained previously are equally suitable for all values of the time : those of this section are useful only for small values of the time, but are much more quickly obtained than the general solutions.

Most of the transforms in Chapters I and II have been found to take the form

$$\bar{y} = \frac{f(p)}{g(p)}, \qquad . \qquad . \qquad . \qquad (22.1)$$

where $g(p)$ is a polynomial in p of degree n, and $f(p)$ a polynomial of lower degree. If we divide numerator and denominator of (22.1) by p^n, and expand in ascending powers of $1/p$ by the Binomial Theorem, we find an expression of the form

$$\bar{y} = \frac{f(p)}{g(p)} = \frac{a_1}{p} + \frac{a_2}{p^2} + \frac{a_3}{p^3} + \cdots \qquad (22.2)$$

and thus $$y = a_1 + a_2 t + a_3\frac{t^2}{2!} + \cdots . \qquad (22.3)$$

by (1.5), and y is obtained as a series of ascending powers of t.

EXAMPLE 1.   In § 9, Example 1, we had

$$I = \frac{E}{L[p^2 + (R/L)p + (1/LC)]}$$

$$= \frac{E}{Lp^2[1 + (R/Lp) + (1/LCp^2)]}$$

$$= \frac{E}{Lp^2}\Big\{1 - \frac{R}{Lp} - \frac{1}{p^2}\Big(\frac{1}{LC} - \frac{R^2}{L^2}\Big) \cdots \Big\}.$$

Thus $I = \dfrac{E}{L}t - \dfrac{ER}{2L^2}t^2 - \dfrac{Et^3}{6L}\Big(\dfrac{1}{LC} - \dfrac{R^2}{L^2}\Big)\cdots$

EXAMPLE 2.   In the problem of § 14, (14.1) gives

$$I = \frac{1 + (1/L_1C_1p^2)}{Rp\{1 + [(C+C_1)/RCC_1p] + [1/L_1C_1p^2] + [1/RCL_1C_1p^3]\}}$$

$$= \frac{1}{Rp}\Big\{1 - \frac{C + C_1}{RCC_1p} + \cdots \Big\}.$$

Therefore, $I = \dfrac{1}{R} - \dfrac{C + C_1}{R^2CC_1}t \cdots$

If we let $p \to \infty$ in (22.2), we find

$$a_1 = \lim_{p \to \infty}\frac{pf(p)}{g(p)} = \lim_{p \to \infty} p\bar{y}(p),$$

and since, by (22.3), a_1 is the value of y when $t \to 0$, we have proved, *for the form* (22.1) *of $\bar{y}$, the following Theorem.*

THEOREM VII.   *If $\bar{y}(p)$ is the transform of $y(t)$, then*

$$\lim_{t \to 0} y(t) = \lim_{p \to \infty} p\bar{y}(p). \qquad . \qquad . \qquad (22.4)$$

This result is in fact true under fairly general conditions.

The following theorem, of the same type as (22.4), may often be used to determine the behaviour of a function for large values of the time:

THEOREM VIII.   *If $\bar{y}(p) = f(p)/g(p)$, where $f(p)$ and $g(p)$ are polynomials in p, the degree of $f(p)$ being less than that of $g(p)$, and if the roots of $g(p) = 0$ are either zero or*

have negative real parts,* then if $y(t)$ is the function whose transform is $\bar{y}(p)$

$$\lim_{t \to \infty} y(t) = \lim_{p \to 0} p\bar{y}(p) \qquad . \qquad . \quad (22.5)$$

and, under the same conditions,

$$\int_0^\infty y(t)dt = \lim_{p \to 0} \bar{y}(p) \qquad . \qquad . \quad (22.6)$$

The result follows immediately from the formulæ of § 2 for expressing $\bar{y}(p)$ in partial fractions. An application of (22.6) has been given in (11.8).

23. Network theorems

Many of the general theorems of steady state network theory have immediate generalizations in terms of Laplace transforms.

Here we mention only Thévenin's theorem because of its importance in simplifying the algebra in complicated networks. A special case of this theorem, which is adequate for many purposes, may be stated as follows :

Suppose AB and CD are two pairs of terminals of a network ; let Z be the generalized impedance of the network looking in at CD with AB short-circuited ; also let $\bar{v}$ be the transform of the open-circuit voltage drop across CD due to voltage $V(t)$ applied for $t > 0$ at AB with zero initial conditions. Then the transform of the current I in an impedance z_0 connected across CD, due to voltage $V(t)$ applied at AB for $t > 0$ with zero initial conditions,

is

$$I = \frac{\bar{v}_{\,.}}{Z + z_0} \qquad . \qquad . \qquad . \quad (23.1)$$

As an example, suppose the network is a filter circuit of m, T sections [Fig. 18 (a)]. In this case the quantity Z defined above is, by (16.10) with $r = 0$,

$$Z = z \sinh \theta \tanh m\theta \quad . \qquad . \quad (23.2)$$

* This Theorem is frequently stated in the literature with no conditions on $\bar{y}(p)$. Clearly it does not hold, for example, if $y(t) = \sin \omega t$.

and the quantity $\bar{v}$ defined above is, by (16.11) with $r = m - 1$,

$$\bar{v} = \text{V sech } m\theta \qquad . \qquad . \qquad (23.3)$$

Thus, by (23.1), the current I in an output impedance z_0 is given by

$$I = \frac{\bar{V}}{z \sinh \theta \sinh m\theta + z_0 \cosh m\theta}.$$

This agrees with (16·7), but the calculation is simpler, since Thévenin's theorem allows the result for any terminal impedance to be deduced from the solutions for open and short-circuited terminations and these are often known and in any case are easy to calculate *ab initio*.

24. *The superposition * theorem*

THEOREM IX. *If $\bar{y}_1(p)$ is the transform of $y_1(t)$, and $\bar{y}_2(p)$ is the transform of $y_2(t)$, then $\bar{y}_1(p)\bar{y}_2(p)$ is the transform of*

$$\int_0^t y_1(t')y_2(t - t')dt' = \int_0^t y_1(t - t')y_2(t')dt'. \quad (24.1)$$

The proof † is not particularly difficult, but is omitted here as it depends on the transformation of double integrals, and involves only routine pure mathematics.

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* Also referred to as the Composition, Convolution, or Faltung Theorem, and sometimes as Borel's or as Duhamel's Theorem.

† C. and J., § 33 ; Doetsch, *loc. cit.*, p. 161. The essential steps, which require justification and some conditions on the functions, are as follows :

$$\bar{y}_1(p)\bar{y}_2(p) = \int_0^\infty e^{-pu}y_1(u)du \int_0^\infty e^{-pv}y_2(v)dv$$

$$= \int\int e^{-p(u+v)}y_1(u)y_2(v)dudv,$$

where the double integral is taken over the quadrant $u > 0$, $v > 0$. In this put $u + v = t$, $v = t'$, then t' can take all values up to t, and t all values up to ∞, and the double integral becomes

$$\int_0^\infty e^{-pt}dt \int_0^t y_1(t - t')y_2(t')dt'.$$

As a first application consider the differential equation

$$(D^2 + a^2)y = f(t), \quad t > 0,$$

to be solved with $y = Dy = 0$, when $t = 0$. The subsidiary equation gives

$$\bar{y} = \frac{\bar{f}(p)}{p^2 + a^2}.$$

Thus by Theorem IX

$$y = \frac{1}{a}\int_0^t f(t') \sin a(t - t')dt'.$$

Clearly, any system of ordinary linear differential equations with constant coefficients and with arbitrary functions on the right hand side can be treated in the same way.

From the present point of view, the chief use of Theorem IX is the following : suppose we require the current I at some place in a network due to voltage $V(t)$ applied at some point with zero initial conditions.* In the usual way we find

$$I = \frac{\bar{V}(p)}{z(p)}, \qquad . \qquad . \qquad . \qquad (24.2)$$

where $z(p)$ is a known function of p. Then if $x(t)$ is the function, found in the usual way, whose transform is $1/z(p)$, Theorem IX gives

$$I = \int_0^t V(t')x(t - t')dt' = \int_0^t V(t - t')x(t')dt' \qquad (24.3)$$

Thus for any applied voltage V, which may be so complicated that we cannot evaluate its transform, or which may be given numerically, for example by an oscillogram, we can always express the current by a definite integral which may be evaluated numerically. In the same way, if the current in any part of a circuit is known, the current in any other part can be determined.

* For non-zero initial conditions there will be additional terms on the right of (24.2) which can be dealt with as in Chapter I.

For example, for voltage V applied to an L, R, C circuit with zero initial conditions, we have as in (9.2)

$$\mathbf{I} = \frac{p\bar{\mathrm{V}}}{\mathrm{L}[(p + \mu)^2 + n^2]},$$

and thus, if $n^2 > 0$,

$$\mathrm{I} = \frac{\mathrm{I}}{n\mathrm{L}}\int_0^t \mathrm{V}(t - t')\{n \cos nt' - \mu \sin nt'\}e^{-\mu t'} \, dt' \quad (24.4)$$

The function $x(t)$ used above whose transform was $1/z(p)$ is, by (24.2), the current due to a unit impulsive applied voltage, $\mathrm{V} = \delta(t)$, $\bar{\mathrm{V}}(p) = 1$. This is an interesting transient in its own right, and the present result shows that if it is known, the current due to any applied voltage, V, may be evaluated as a definite integral. The other simple and interesting transient in the circuit is the current I_1 due to unit applied voltage, $\mathrm{V} = 1$, which, by (24.2) has the transform

$$\bar{\mathrm{I}}_1 = \frac{1}{pz(p)}. \qquad . \qquad . \qquad . \qquad (24.5)$$

If I_1 has been found as a function of t from (24.5), we may express the current due to any applied voltage V as an integral involving I_1, by writing (24.2) in the form

$$\bar{\mathrm{I}} = p \cdot \bar{\mathrm{V}}(p) \cdot \frac{1}{pz(p)}. \qquad . \qquad . \qquad (24.6)$$

Now, by Theorem IX, the function whose transform is $\bar{\mathrm{V}}(p)/[pz(p)]$ is

$$\int_0^t \mathrm{I}_1(t')\mathrm{V}(t - t')dt'. \qquad . \qquad . \qquad (24.7)$$

Then it follows from (24.6) and Theorem III that

$$\mathrm{I} = \frac{d}{dt}\int_0^t \mathrm{I}_1(t')\mathrm{V}(t - t')dt', \qquad . \qquad . \qquad (24.8)$$

since (24.7) vanishes when $t = 0$. Clearly various transformations of (24.8) are possible.

Finally, it should be remarked that the importance of

Theorem IX is in the study of difficult or arbitrary functions, or functions given numerically by oscillograms; if V is a simple exponential or trigonometric function, it is simpler to evaluate I directly by the methods used earlier than to evaluate the integrals (24.3) or (24.8).

25. *The inversion theorem*

This is a general theorem * which gives $y(t)$ from $\bar{y}(p)$, subject, of course, to conditions on $y(t)$ or $\bar{y}(p)$. Omitting these conditions, it states that if

$$\bar{y}(p) = \int_0^\infty e^{-pt}y(t)dt, \quad \mathbf{R}(p) > 0, \qquad (25\cdot1)$$

then
$$y(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\bar{y}(\lambda)d\lambda, \quad \gamma > 0. \qquad (25\cdot2)$$

In (25.2), λ is a complex variable, and the integral is to be interpreted and evaluated by the theory of functions of a complex variable. If this theory, and in particular the calculus of residues, is known, (25.2) provides an important method for finding the function which has a given transform. For the problems of Chapters I and II which led to ordinary linear differential equations, this gives an alternative method for finding $y(t)$, which, however, is no shorter than those used earlier, so that for such problems there is no necessity to learn the technique of complex variable. For a thorough study of the applications of the method to partial differential equations, the use of the inversion theorem is necessary.

From the point of view of the historical development of the subject the correspondence between (25.1) and (25.2) is of great importance. Carson † regarded (25.1) as an integral equation from which the solution of the problem could be obtained. Bromwich, and to a certain extent

* C. and J., §§ 29, 30. Doetsch, *loc. cit.*, Kap. 6. The result is sometimes called the Fourier-Mellin Theorem. It is related to Fourier's and Mellin's inversion formulæ, cf. Doetsch, *loc. cit.*
† *Electric Circuit Theory and Operational Calculus* (McGraw Hill, 1926).

Wagner, Jeffreys, and McLachlan, regard an integral of the type (25.2), with $\bar{y}(\lambda)$ found from the usual subsidiary equation, as the solution of the problem. The Laplace transformation and its inversion theorem provides the link between these two ideas ; this point of view is largely due to Doetsch.

26. *The connection with Fourier's integral theorem and Parseval's theorem*

Fourier's integral theorem * states that, subject to conditions on $y(t)$, if ω is real and

$$Y(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} y(t) dt, \qquad . \qquad . \qquad (26.1)$$

then

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} Y(\omega) d\omega. \qquad . \qquad (26.2)$$

The simplest of the essential conditions on $y(t)$ is that

$$\int_{-\infty}^{\infty} |y(t)| \, dt \quad \text{be convergent.} \qquad . \qquad (26.3)$$

Parseval's theorem† states that, roughly provided these integrals exist,

$$\int_{-\infty}^{\infty} |y(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(\omega)|^2 \, d\omega. \qquad . \qquad (26.4)$$

For comparison with the problems previously discussed, suppose that $y(t)$ is real, and is zero ‡ when $t < 0$. Then (26.1) and (26.2) become :

* For a proof see Carslaw, Fourier Series and Integrals, edn. 3, § 119 ; the result (26.2), (26.1) follows from his on using the exponential form for the cosine. See also Titchmarsh, *Theory of Fourier Integrals*, §§ 1.1, 1.2.

† Titchmarsh, *loc. cit.*, § 2.1. The theorem is often referred to as Rayleigh's theorem, and its use in physical applications is due to him [*Phil. Mag.*, **27** (1889), 466].

‡ It is frequently stated that the Laplace transformation method contemplates only functions of this type. In the method as developed in Chapters I and II the problem was to find the solution of a differential equation for $t > 0$, which satisfied certain conditions at $t = 0$, and in fact the solutions found would also hold

if $\qquad Y(\omega) = \int_0^\infty e^{i\omega t} y(t) dt,$ $\qquad$. $\qquad$. $\qquad$. $\qquad$. $\qquad$ (26.5)

then * $\qquad \dfrac{1}{\pi} R \int_0^\infty e^{-i\omega t} Y(\omega) d\omega = y(t),$ if $t > 0$

and $\qquad \dfrac{1}{\pi} R \int_0^\infty e^{-i\omega t} Y(\omega) d\omega = 0,$ if $t < 0$ $\qquad$, (26.6)

provided $\qquad \int_0^\infty |y(t)| \, dt$ is convergent. $\qquad$. (26.7)

To deduce (26.6) from (26.2) in this case, we notice that the right-hand side of the latter is equal to

$$\frac{1}{2\pi}\int_0^\infty e^{-i\omega t} Y(\omega) d\omega + \frac{1}{2\pi}\int_0^\infty e^{i\omega t} Y(-\omega) d\omega,$$

and, since $y(t)$ is real, $Y(-\omega)$ is the conjugate of $Y(\omega)$, and the result (26.6) follows.

Also, in this case (26.4) becomes

$$\int_0^\infty |y(t)|^2 \, dt = \frac{1}{\pi}\int_0^\infty |Y(\omega)|^2 \, d\omega. \qquad . \quad (26.8)$$

$Y(\omega)$ in (26.5) is called the Fourier transform of $y(t)$. Since (26.5) is precisely the same as the equation defining the Laplace transform $\bar{y}(p)$ of $y(t)$, with p replaced by $(-i\omega)$, it follows that, if $Y(\omega)$ *exists*,

$$Y(\omega) = \bar{y}(-i\omega), \qquad . \qquad . \qquad (26.9)$$

Thus the Fourier transform is the degenerate form of the Laplace transform if the latter exists when p is pure imaginary, also (26.6) is the degenerate form of (25.2) if

for $t < 0$; for example, the solution of $(D + 1)y = 0$ with $y = 1$ when $t = 0$ is found to be $y = e^{-t}$ for either $t > 0$ or $t < 0$, though we are only interested in the former case. In this section and in § 25 there is a difference in principle (though not in the results obtained for $t > 0$) since a solution is sought which is zero for $t < 0$; for example the solution of the differential equation above, found from (25.2), would be $y(t) = 0$, $t < 0$; $y(t) = \frac{1}{2}$, $t = 0$; $y(t) = e^{-t}$, $t > 0$.

* R is written for " the real part of ".

γ can be taken zero. The Laplace transform exists for many functions for which the Fourier transform does not (for example, 1, e^t or $\sin t$) but, if the latter does exist, it can be found from (26.9) and previous results.

For example,* if $\quad y(t) = e^{-\alpha t}\sin \beta t \qquad . \qquad .$ (26.10)

so that $\qquad\qquad \bar{y}(p) = \dfrac{\beta}{(p + \alpha)^2 + \beta^2}, \quad .$

$$Y(\omega) = \bar{y}(-i\omega) = \frac{\beta}{(\alpha - i\omega)^2 + \beta^2} = \frac{\beta}{(\alpha^2 + \beta^2 - \omega^2) - 2i\alpha\omega}.$$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad . \qquad .$ (26.11)

In this case (26.8) gives

$$\int_0^\infty e^{-2\alpha t} \sin^2 \beta t\, dt = \frac{\beta^2}{\pi}\int_0^\infty \frac{d\omega}{(\alpha^2 + \beta^2 - \omega^2)^2 + 4\alpha^2\omega^2}.$$ (26.12)

From the point of view of electric circuit theory the great importance of Fourier's integral theorem lies in its physical interpretation. The Laplace transform has no obvious physical significance ; it is a purely mathematical device for finding a solution—this is not altogether a disadvantage as it implies that any difficulties can be formulated and treated mathematically, while it was on points of interpretation that the old operational method was obscure. On the other hand (26.6) expresses $y(t)$ as a combination (that is the integral) of harmonic terms $e^{-i\omega t}$, each with its own amplitude $Y(\omega)d\omega$, in the same way that a periodic quantity is expressed by Fourier's theorem as a sum of fundamental and harmonics, each with its own amplitude.

Now suppose a voltage $y(t)$ is applied to a circuit with zero initial conditions, and the current I in any branch of it is calculated, then, proceeding as in Chapter II we find

$$\bar{I}(p) = \frac{\bar{y}(p)}{z(p)} \qquad . \qquad . \qquad .$$ (26.13)

* Fourier transforms of many functions of this type, with numerical illustrations and applications, are given in Burch and Bloemsma, *Phil. Mag.* (6), **49** (1925), 480.

where $z(p)$ is known and depends on the circuit. Then if $y(t)$ has a Fourier transform, this will be $\bar{y}(-i\omega)$, by (26.9), and (26.13) gives for the Fourier transform of $I(t)$

$$I(-i\omega) = \frac{\bar{y}(-i\omega)}{z(-i\omega)}. \qquad . \qquad . \quad (26.14)$$

Thus, by (26·6)

$$I(t) = \frac{1}{\pi}R\int_0^\infty e^{-i\omega t}\frac{\bar{y}(-i\omega)}{z(-i\omega)}d\omega, \quad t > 0. \quad (26.15)$$

This expression has a simple physical interpretation, since $e^{-i\omega t}/z(-i\omega)$ is the steady state current in the circuit due to voltage $e^{-i\omega t}$. Thus (26.15) may be interpreted by the statement that the current due to voltage $y(t)$ is obtained by superposing the effects of all the component frequencies of $y(t)$, each with its appropriate amplitude $\bar{y}(-i\omega)d\omega$. That is, we have an extension of the results of steady state alternating current theory to non-periodic applied voltages, although, for the solution of specific problems, this is inferior to the Laplace transformation methods described earlier.

One application of (26.14), however, is new and of great importance. Applying Parseval's theorem (26.8) to (26.14) gives

$$\int_0^\infty I^2 dt = \frac{1}{\pi}\int_0^\infty \left|\frac{\bar{y}(-i\omega)}{z(-i\omega)}\right|^2 d\omega. \qquad . \quad (26.16)$$

The quantity on the left gives the energy absorbed in the branch of the circuit, alternatively it is the total response of a square law detector, which is of considerable practical interest, and (26·16) gives this as a real infinite integral without having actually to evaluate the current.

For example, if voltage $e^{-\alpha t}$ is applied at $t = 0$ to an inductive resistance R, L we have

$$\int_0^\infty I^2 dt = \frac{1}{\pi}\int_0^\infty \frac{d\omega}{|(R - Li\omega)(\alpha - i\omega)|^2}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{d\omega}{L^2(\omega^2 + \beta^2)(\omega^2 + \alpha^2)} \qquad \textit{[continued overleaf}$$

7

$$= \frac{1}{\pi L^2(\alpha^2 - \beta^2)}\left[\frac{1}{\beta}\tan^{-1}\frac{\omega}{\beta} - \frac{1}{\alpha}\tan^{-1}\frac{\omega}{\alpha}\right]_0^\infty$$

$$= \frac{1}{2L^2\alpha\beta(\alpha + \beta)},$$

where $\beta = R/L$.

In more complicated problems the integral (26.16) may be evaluated numerically, and approximations can be introduced, for example, with a selective circuit, $|z(i\omega)|$ is very large except in a certain range of values of ω, and (26.16) need be evaluated only over this range.

EXAMPLES ON CHAPTER III

1. If $y(t)$ is $1, 2, 3, \ldots$ in the regions $0 < t < T$, $T < t < 2T$, $2T < t < 3T$, and so on (a " staircase " function)

$$\bar{y}(p) = \frac{1}{p(1 - e^{-Tp})}.$$

2. If $y(t) = 1$, $0 < t < t_1$; $y(t) = 0$, $t_1 < t < T$; and this is repeated with period T (a repeated pulse, or the difference of two staircase functions)

$$\bar{y}(p) = \frac{1 - e^{-pt_1}}{p(1 - e^{-pT})}.$$

3. If $y(t) = |\sin \omega t|$, (full wave rectified alternating voltage)

$$\bar{y}(p) = \frac{\omega(1 + e^{-p\pi/\omega})}{(p^2 + \omega^2)(1 - e^{-p\pi/\omega})}.$$

4. A uniform beam is clamped horizontally at both ends. It carries a distributed load w (constant) per unit length in $0 < x < \frac{1}{2}l$ and zero in $\frac{1}{2}l < x < l$. Show that the deflection at any point is

$$-\frac{13wlx^3}{192EI} + \frac{11wl^2x^2}{384EI} + \frac{wx^4}{24EI} - \frac{w(x - \frac{1}{2}l)^4}{24EI}H(x - \frac{1}{2}l).$$

5. If the square-wave voltage of § 21, Ex. 3, is applied at $t = 0$ to an L, R, C circuit with zero initial current and charge, show that the current in the circuit at time $t = 2sT + t'$, where s is an integer and $t' < T$, is

$$\frac{2E}{nL}e^{-\mu t'}\sum_{r=0}^{2s-1}(-1)^r e^{-\mu rT}\sin n(rT + t') + \frac{E}{nL}e^{-\mu t}\sin nt,$$

using the notation (9.3) and assuming $n^2 > 0$. Deduce that in the steady state due to a square wave voltage, the current at time t' after the beginning of a cycle is

$$\frac{2E}{nL} \frac{e^{-\mu t'} \sin nt' - e^{-\mu(T+t')} \sin n(T - t')}{1 + 2e^{-\mu T} \cos nT + e^{-2\mu T}}.$$

6. In the problem of § 11, Ex. 2, show that for small values of the time

$$I = -\frac{E}{L} \left\{ t - \frac{n^2 t^3}{6} + \frac{n^4 t^5}{60} \cdots \right\}.$$

7. In the circuit of Fig. 7, an unknown voltage is applied at S at $t = 0$ with zero initial currents and charges. If the voltage drop across the first condenser is $v(t)$, show that the voltage drop across the second condenser is

$$n \int_0^t v(t') \sin n(t - t') dt'.$$

8. If $y(t) = (e^{-\alpha t} - e^{-\beta t})$, $t > 0$, show that its Fourier transform $Y(\omega)$ is

$$\frac{\beta - \alpha}{(\alpha - i\omega)(\beta - i\omega)}.$$

Show that the total energy absorbed by an inductive resistance L, R from this voltage is

$$\frac{R(\beta - \alpha)^2 (\alpha + \beta + \mu)}{2L^2 \alpha \beta \mu (\alpha + \beta)(\beta + \mu)(\mu + \alpha)},$$

where $\mu = R/L$.

PARTIAL DIFFERENTIAL EQUATIONS

27. *Introductory*

In the preceding chapters it has been found that problems leading to ordinary linear differential equations could be solved with only the elementary apparatus of §§ 1-3. In the study of partial differential equations it will appear that it is still easy to find the Laplace transform of the solution, but, since this usually involves more complicated functions of p than those considered in Chapter I, it is rather more difficult to derive the solution from it. The Table of Transforms can of course be enlarged, but the most power- ful general method is the application of the inversion theorem, § 25, combined with the use of contour integration to evaluate the complex integral which occurs in (25.2). For a complete study of the subject, particularly in its theoretical aspects, a knowledge of this technique is essential; but it is possible to go far without it, for example most results on finite transmission lines, and some on infinite lines, can be obtained, in some cases rigorously, and in others with the proviso that for a complete treatment the complex variable would have to be used, although the actual process of writing down the solution would be precisely the same as that used here.

The differential equations for a uniform electric trans- mission line will be found in § 28, and, for shortness, only these will be studied, although many of the problems considered are problems in wave motion or conduction of heat in one dimension, with a slightly altered notation.

28. *The differential equations of the uniform transmission line*

We suppose the line to have resistance R, inductance L, capacity C, and leakage conductance G, per unit length.

Also let I be the current in the direction of increasing x at the point x of the line, and let V be the voltage drop across the line at this point. I and V will be functions of both distance x along the line and time t; to emphasize this they will sometimes be written $I(x, t)$ and $V(x, t)$.

To find the differential equations satisfied by the current and voltage in the line, consider the points A at x, and B at $x + \delta x$. The current and voltage at A will be I and V, and at B they will be $I + \dfrac{\partial I}{\partial x}\delta x$ and $V + \dfrac{\partial V}{\partial x}\delta x$, respectively.

The portion of the line between A and B can be represented
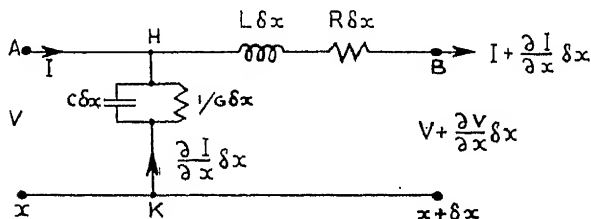

FIG. 28.

by the equivalent circuit of Fig. 28. Here the voltage drop from A to B over the series portion of the line is $-\dfrac{\partial V}{\partial x}\delta x$, and the current in it is $I + \dfrac{\partial I}{\partial x}\delta x$; neglecting terms in $(\delta x)^2$ this gives

$$\left(L\frac{\partial I}{\partial t} + RI\right)\delta x = -\frac{\partial V}{\partial x}\delta x \qquad . \quad (28.1)$$

Also the voltage drop over the parallel portion HK of the line is V, and the current in it is $-\dfrac{\partial I}{\partial x}\delta x$, thus

$$\left(C\frac{\partial V}{\partial t} + GV\right)\delta x = -\frac{\partial I}{\partial x}\delta x. \qquad . \quad (28.2)$$

From (28.1) and (28.2) it follows that the differential equations for I and V are

$$L\frac{\partial I}{\partial t} + RI = -\frac{\partial V}{\partial x} \qquad . \qquad . \qquad (28.3)$$

$$C\frac{\partial V}{\partial t} + GV = -\frac{\partial I}{\partial x} \qquad . \qquad . \qquad (28.4)$$

These are a pair of simultaneous partial differential equations for I and V. We wish to solve them for $t > 0$ with given initial values of I and V, which we shall write $\mathring{I}(x)$ and $\mathring{V}(x)$ to emphasize that they are functions of x. Explicitly these initial conditions are

$$I(x, t) \to \mathring{I}(x), \text{ as } t \to 0 \text{ for fixed } x, \quad . \quad (28.5)$$
$$V(x, t) \to \mathring{V}(x), \text{ as } t \to 0 \text{ for fixed } x. \quad . \quad (28.6)$$

There are also boundary conditions to be satisfied at the ends of the line; these depend on the particular problem. For example, if the line is short-circuited at $x = a$, we must have $V = 0$ for $x = a$ and all $t > 0$; if there is open circuit at $x = a$, the condition is $I = 0$ for $x = a$ and all $t > 0$; if the voltage is prescribed at $x = a$, $V(a, t)$ must be this prescribed function of t for $t > 0$; if the line is joined to some other impedance at $x = a$, there will be a condition involving both the current and voltage at $x = a$.

29. *The solution of partial differential equations by the Laplace transformation*

We consider first the system of equations (28.3), (28.4) which have to be solved for $t > 0$ with initial conditions (28.5) and (28.6), and with boundary conditions to be specified. Proceeding * precisely as in Chapter I, § 4, we multiply the equations by e^{-pt} and integrate with respect to t from 0 to ∞. Then, just as in the proof of Theorem III,

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* As in Chapter I assumptions about the function I are implied, and, strictly, the solution should be verified.

$$\int_0^\infty e^{-pt}\frac{\partial I(x,t)}{\partial t}dt = \left[e^{-pt}I(x,t)\right]_{t=0}^{t=\infty} + p\int_0^\infty e^{-pt}I(x,t)dt$$

$$= -\mathring{I}(x) + p\bar{I} \quad . \qquad . \qquad . \qquad (29.1)$$

where $\bar{I}$ is now a function of x and p, and, when we wish to emphasize this, it is written $\bar{I}(x,p)$.

Also $\quad \int_0^\infty e^{-pt}\frac{\partial I(x,t)}{\partial x}dt = \frac{d}{dx}\int_0^\infty e^{-pt}I(x,t)dt = \frac{d\bar{I}}{dx}$ $\quad$ (29.2)

where we have assumed that the orders of integration, and of differentiation with respect to the parameter x, may be interchanged.

Using these results, and the corresponding ones for V, we obtain from (28.3) and (28.4) and the initial conditions, the " subsidiary equations "

$$(Lp + R)\bar{I}(x,p) = -\frac{d\bar{V}(x,p)}{dx} + L\mathring{I}(x) \qquad (29.3)$$

$$(Cp + G)\bar{V}(x,p) = -\frac{d\bar{I}(x,p)}{dx} + C\mathring{V}(x) \qquad (29.4)$$

Eliminating $\bar{I}$ gives the ordinary differential equation for $\bar{V}$,

$$\frac{d^2\bar{V}}{dx^2} - q^2\bar{V} = L\frac{d\mathring{I}}{dx} - C(Lp + R)\mathring{V}, \quad . \quad (29.5)$$

where $\qquad\qquad q^2 = (Lp + R)(Cp + G) \quad . \qquad . \quad (29.6)$

When $\bar{V}$ has been found, $\bar{I}$ is given by (29.3), that is, by

$$\bar{I} = -\frac{1}{Lp + R}\frac{d\bar{V}}{dx} + \frac{L\mathring{I}(x)}{Lp + R}. \qquad . \quad (29.7)$$

It appears that the effect of the Laplace transformation with respect to t is to reduce a partial differential equation in x and t to an ordinary differential equation in x. This has to be solved with boundary conditions which are the Laplace transforms of those originally prescribed—they are best written down as they arise in specific problems.

The quantity q which appears in (29.6) is, in the general

case, the square root of a quadratic function of p, but in three important cases it takes a simpler form. These are:
(i) The " lossless line ", $R = G = o$, for which

$$q = p/c, \text{ where } c = (LC)^{-\frac{1}{2}} . \qquad . \quad (29.8)$$

(ii) Heaviside's " distortionless line " in which

$$\frac{R}{L} = \frac{G}{C} = \rho,$$

and thus $\cdot \quad q = (p + \rho)/c, \text{ where } c = (LC)^{-\frac{1}{2}}. \quad . \quad (29.9)$
(iii) The ideal submarine cable, $L = G = o$, for which

$$q = \sqrt{(RCp)}. \qquad . \qquad . \quad (29.10)$$

Another partial differential equation of importance is the equation of linear flow of heat (v temperature, κ diffusivity)

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = o, \qquad . \qquad . \quad (29.11)$$

to be solved with $v = \overset{\circ}{v}(x)$, when $t = o$. The subsidiary equation, found as above, is

$$\frac{d^2\bar{v}}{dx^2} - \frac{p}{\kappa}\bar{v} = -\frac{1}{\kappa}\overset{\circ}{v}(x) . \qquad . \qquad . \quad (29.12)$$

and thus is of the same type as (29.5) in the case (29.10) of $L = G = o$.

The equation of wave motion in one dimension is

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = o, \qquad . \qquad . \quad (29.13)$$

where y is the displacement. This has to be solved with

$$y = \overset{\circ}{y}(x), \quad \frac{\partial y}{\partial t} = \overset{\bullet\circ}{Y}(x), \quad \text{when } t = o.$$

The subsidiary equation is

$$\frac{d^2\bar{y}}{dx^2} - \frac{p^2}{c^2}\bar{y} = -\frac{p}{c^2}\overset{\circ}{y}(x) - \frac{1}{c^2}\overset{\circ}{Y}(x), \qquad . \quad (29.14)$$

which is of the same type as (29.5) for the case (29.8) of $R = G = 0$.

When the subsidiary equation (29.5) with its boundary conditions has been solved,* we know the transform $\bar{V}(x, p)$ of the solution $V(x, t)$. For problems on semi-infinite lines we rely on the Table of Transforms to find the solution from its transform. For problems on finite lines two methods are available : (i) an extension of the expansion theorem (2.8), which gives the solution in the form of a trigonometric series ; or (ii) an expansion in terms of the solutions for the semi-infinite line, which has a physical interpretation in terms of direct and multiply reflected waves. Both these methods will be used, treating the latter first.

30. *The semi-infinite line*

We consider the problem of the semi-infinite line, $x > 0$, with zero initial current and charge. First, suppose the end $x = 0$ to be maintained at constant voltage E for $t > 0$.

In this case the subsidiary equation (29.5) becomes

$$\frac{d^2\bar{V}}{dx^2} - q^2\bar{V} = 0, \quad x > 0. \quad . \quad . \quad (30.1)$$

The boundary conditions to be satisfied at the ends of the line are

$$V = E, \quad x = 0, \quad t > 0, \quad . \quad . \quad (30.2)$$

$$V \text{ to be finite as } x \to \infty \quad . \quad . \quad (30.3)$$

The transforms of (30.2) and (30.3) are

$$\bar{V} = E/p, \quad x = 0, \quad . \quad . \quad . \quad (30.4)$$

$$\bar{V} \text{ to be finite as } x \to \infty. \quad . \quad . \quad (30.5)$$

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* We could, as in § 6, solve it by the Laplace transformation method, but, for the simple problems arising here, it is just as easy to write down the particular integral and complementary function in the usual way.

The general solution of (30.1) is
$$\bar{V} = Ae^{-qx} + Be^{qx},$$
and (30.5) requires B = o, and then (30.4) gives A = E/p. Thus, using the value (29.6) of q,
$$\bar{V} = \frac{E}{p}e^{-qx} = \frac{E}{p}e^{-x\sqrt{(Lp+R)(Cp+G)}} \qquad . \quad (30.6)$$

In the general case the transform involved in (30.6) is quite different from anything previously encountered, and in fact V turns out, after a difficult calculation which will not be given here, to be a complicated integral involving Bessel functions (cf. § 35). In the important special cases mentioned in § 29, however, V can be evaluated simply.

For the lossless line, R = G = o (30.6) becomes
$$\bar{V} = \frac{E}{p}e^{-px/c}, \qquad . \qquad . \qquad . \quad (30.7)$$
where
$$c = (LC)^{-\frac{1}{2}}. \qquad . \qquad . \qquad . \quad (30.8)$$

Then it follows from (21.1), with $y(t)$ = E, that
$$V = EH[t - (x/c)], \qquad . \qquad . \quad (30.9)$$

that is : V is zero up to the time x/c, at which a wave travelling with velocity c from the origin would arrive at the point x, and V has the constant value E at this point for greater times.

For the distortionless line (29.9) we have
$$\bar{V} = \frac{E}{p}e^{-(p+\rho)x/c}, \qquad . \qquad . \qquad . \quad (30.10)$$
and
$$V = Ee^{-\rho x/c}H(t - x/c) \qquad . \qquad . \quad \cdot(30.11)$$

In this case the disturbance is still propagated with velocity c, but there is attenuation as we move along the line.

Some problems on the submarine cable, L = G = o, will be studied in § 35.

Suppose, now, that instead of applying a constant

voltage E at $x = 0$, we apply a sinusoidal voltage E sin ωt. The only modification is that in place of (30.6) we have

$$\bar{V} = \frac{E\omega}{p^2 + \omega^2}e^{-qx}. \qquad . \qquad . \quad (30.12)$$

In the case of the distortionless line, $q = (p + \rho)/c$, this becomes

$$\bar{V} = \frac{E\omega}{p^2 + \omega^2}e^{-(p+\rho)x/c}, \quad . \qquad . \quad (30.13)$$

and $\qquad$ $V = Ee^{-\rho x/c} \sin \omega\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right). \quad (30.14)$

Still more generally, for any applied voltage, $f(t)$, we have in place of (30.6)

$$\bar{V} = \bar{f}(p)e^{-qx}. \qquad . \qquad . \quad (30.15)$$

For the distortionless line this becomes

$$\bar{V} = \bar{f}(p)e^{-(p+\rho)x/c}, \qquad . \quad (30.16)$$

and, again using (21.1),

$$V = e^{-\rho x/c}f\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right). \quad . \quad (30.17)$$

This may be interpreted by the statement that the voltage at x is zero up to time x/c, and subsequently follows that at $x = 0$ with a time lag of x/c, and a reduction in magnitude by the factor $e^{-\rho x/c}$.

In the next section the application to problems on finite lines of the solutions found above will be studied.

31. *The travelling wave solutions for the finite transmission line*

As an example of the method we consider the problem of the finite transmission line $0 < x < l$, with zero initial current and charge. The end $x = 0$ is to be earthed, and the end $x = l$ maintained at constant voltage E for $t > 0$.

The subsidiary equation (29·5) becomes in this case

$$\frac{d^2\bar{V}}{dx^2} - q^2\bar{V} = o, \quad o < x < l. \qquad . \quad (31.1)$$

The boundary conditions are

$$V = E, \quad x = l, \quad t > o, \qquad . \qquad . \quad (31.2)$$

$$V = o, \quad x = o, \quad t > o. \qquad . \qquad . \quad (31.3)$$

The transforms of (31.2) and (31.3) are

$$\bar{V} = E/p, \quad x = l, \qquad . \qquad . \quad (31.4)$$

$$\bar{V} = o, \qquad x = o. \qquad . \qquad . \quad (31.5)$$

The solution of (31.1) which satisfies (31.4) and (31.5) is

$$\bar{V} = \frac{E}{p} \cdot \frac{\sinh qx}{\sinh ql}. \qquad . \qquad . \quad (31.6)$$

Here q is defined by (29.6) so that $\bar{V}$ is a complicated function of p. One method of proceeding is suggested by the device of expanding in a series of negative exponentials which was used in § 21, Example 6. We write (31.6) in a form involving negative exponentials, and expand the denominator by the binomial theorem. This gives

$$\bar{V} = \frac{E}{p} \cdot \frac{e^{qx}(\mathbf{1} - e^{-2qx})}{e^{-ql}(\mathbf{1} - e^{-2ql})}$$

$$= \frac{E}{p} e^{-q(l-x)}(\mathbf{1} - e^{-2qx})(\mathbf{1} + e^{-2ql} + e^{-4ql} + \ldots)$$

$$= \frac{E}{p} \Big\{ e^{-q(l-x)} - e^{-q(l+x)} + e^{-q(3l-x)} - e^{-q(3l+x)} + \ldots \Big\}. \quad (31.7)$$

In the special case of the lossless line, for which, by (29.8), $q = p/c$, (31.7) becomes

$$\bar{V} = \frac{E}{p} \Big\{ e^{-p(l-x)/c} - e^{-p(l+x)/c} + e^{-p(3l-x)/c} - \ldots \Big\}. \quad (31.8)$$

And, using (21.1)

$$V = E\left\{H\left(t - \frac{l-x}{c}\right) - H\left(t - \frac{l+x}{c}\right) + H\left(t - \frac{3l-x}{c}\right) - \ldots\right\}.$$
$$\text{. (31.9)}$$

The graph of the function (31.9) is shown in Fig. 29. The voltage at x is zero up to time $(l-x)/c$, at which a wave travelling direct from the end $x = l$ would reach the point x. The voltage then has the constant value E up to the time $(l+x)/c$, at which a wave travelling from the end $x = l$ and reflected back from the end $x = o$ would arrive. From this time up to the time of arrival of a twice reflected wave, it has the value zero, and so on.
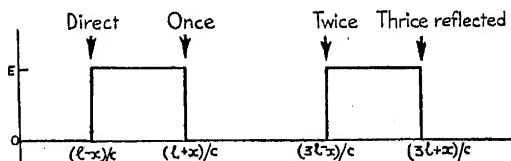

FIG. 29.

Returning to the general expression (31.7), it is seen that the terms of this are all of the form $(1/p)e^{-aq}$, which appeared in (30.6) in the corresponding problem for the semi-infinite line. Thus, if the solution for this case is known, the solution for a finite line can be expressed in terms of it.

As a second example, we consider a lossless line of length l, initially charged to constant voltage E. At $t = o$ the end $x = o$ is earthed, the end $x = l$ being left insulated.

In this case the subsidiary equation (29.5) becomes, writing $c = (LC)^{-\frac{1}{2}}$,

$$\frac{d^2\bar{V}}{dx^2} - \frac{p^2}{c^2}\bar{V} = -\frac{pE}{c^2}, \qquad . \qquad . \quad (31.10)$$

which has to be solved with

$$\bar{V} = 0, \quad x = 0, \qquad . \qquad . \quad (31.11)$$

$$\bar{I} = 0, \quad x = l. \qquad . \qquad . \quad (31.12)$$

Also it follows from (29.7) that

$$\bar{I} = -\frac{1}{Lp}\frac{d\bar{V}}{dx}. \qquad . \qquad . \quad (31.13)$$

The solution of (31.10) which satisfies (31.11) and (31.12) is

$$\bar{V} = \frac{E}{p} - \frac{E}{p} \cdot \frac{\cosh p(l-x)/c}{\cosh pl/c} \qquad . \qquad . \qquad . \qquad . \quad (31.14)$$

$$= \frac{E}{p}\left\{1 - \frac{e^{-px/c}[1 + e^{-2p(l-x)/c}]}{1 + e^{-2pl/c}}\right\}$$

$$= \frac{E}{p}\{1 - e^{-px/c} - e^{-p(2l-x)/c} + e^{-p(2l+x)/c} + \ldots\}.$$

Thus $\dfrac{V}{E} = 1 - H\left(t - \dfrac{x}{c}\right) - H\left(t - \dfrac{2l-x}{c}\right)$

$$+ H\left(t - \frac{2l+x}{c}\right) + \ldots \quad (31.15)$$

Also by (31.13)

$$\bar{I} = -\frac{E}{p}\sqrt{\left(\frac{C}{L}\right)}\frac{\sinh p(l-x)/c}{\cosh pl/c}. \quad . \quad (31.16)$$

The most interesting quantity is the current at $x = 0$. At this point

$$\bar{I} = -\frac{E}{p}\sqrt{\left(\frac{C}{L}\right)}\frac{1 - e^{-2pl/c}}{1 + e^{-2pl/c}}$$

$$= -\frac{E}{p}\sqrt{\left(\frac{C}{L}\right)}\{1 - 2e^{-2pl/c} + 2e^{-4pl/c} - \ldots\}.$$

$$I = -E\sqrt{\left(\frac{C}{L}\right)}\left\{1 - 2H\left(t - \frac{2l}{c}\right) + 2H\left(t - \frac{4l}{c}\right) - \ldots\right\},$$

$$(31.17)$$

that is, a square wave of amplitude $E\sqrt{(C/L)}$ and period $(4l/c)$.

32. *The finite line with terminal impedances*

Suppose that at $x = l$, where the voltage is V_l and the current is I_l, the line is terminated by impedance z_l. Then, for zero initial conditions in this impedance, the boundary condition at $x = l$ is just the subsidiary equation for the impedance z_l, namely

$$z_l \bar{I}_l = \bar{V}_l . \qquad . \qquad (32.1)$$

FIG. 30.

Similarly, if voltage v is applied through impedance z_0 to the line at $x = 0$; writing V_0 for the voltage at the point $x = 0$ of the line and I_0 for the current there, the boundary condition at $x = 0$, for zero initial conditions in the impedance z_0 is

$$z_0 \bar{I}_0 = \bar{v} - \bar{V}_0. \qquad . \qquad . \qquad (32.2)$$

As an example, suppose that a lossless line of length l is initially charged to voltage E, and that at $t = 0$ it is discharged through a resistance R_0 at $x = 0$, the end $x = l$ being insulated. In this case the subsidiary equation (29.5) becomes

$$\frac{d^2\bar{V}}{dx^2} - \frac{p^2}{c^2}\bar{V} = -\frac{p\mathrm{E}}{c^2}, \quad 0 < x < l, \qquad . \qquad (32.3)$$

where $c = (LC)^{-\frac{1}{2}}$. At $x = l$, we have

$$\bar{I} = 0, \quad x = l, . \qquad . \qquad . \qquad (32.4)$$

and at $x = 0$ the boundary condition (32.2) is

$$R_0 \bar{I}_0 = -\bar{V}_0. \qquad . \qquad . \qquad (32.5)$$

Also we know from (29.7) that

$$\bar{I} = -\frac{1}{Lp}\frac{d\bar{V}}{dx}, \qquad . \qquad . \qquad (32.6)$$

and thus (32.5) becomes

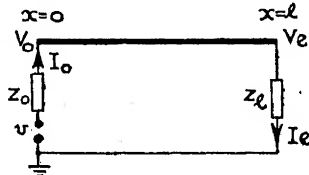$$-\frac{R_0}{Lp}\frac{d\bar{V}}{dx} + \bar{V} = 0, \quad x = 0. \qquad . \qquad (32.7)$$

The general solution of (32.3) which satisfies (32.4) is

$$\bar{V} = \frac{E}{p} + A \cosh p(l - x)/c. \qquad . \quad (32.8)$$

Substituting this in (32.7) gives

$$A \frac{R_0}{Lc} \sinh \frac{pl}{c} + A \cosh \frac{pl}{c} + \frac{E}{p} = 0.$$

Using this value of A in (32.8), we find

$$\bar{V} = \frac{E}{p} \Big\{ 1 - \frac{Lc \cosh p(l - x)/c}{R_0 \sinh (pl/c) + Lc \cosh (pl/c)} \Big\}. \quad (32.9)$$

The current is obtained from (32.6) which gives

$$\bar{I} = - \frac{E \sinh p(l-x)/c}{p\{R_0 \sinh (pl/c) + R_1 \cosh (pl/c)\}}, \quad (32.10)$$

where $R_1 = Lc = \sqrt{(L/C)}$. For shortness we evaluate only the current at $x = 0$. In this case, expressing (32.10) in negative exponentials, and writing $\alpha = (R_1 - R_0)/(R_1 + R_0)$, we find

$$\bar{I} = - \frac{E(1 - e^{-2pl/c})}{p(R_1 + R_0)(1 + \alpha e^{-2pl/c})}$$

$$= - \frac{E}{(R_1 + R_0)p} \Big\{ 1 + (\alpha + 1) \sum_{n=1}^{\infty} (-1)^n \alpha^{n-1} e^{-2npl/c} \Big\}.$$

$$(32.11)$$

Therefore

$$I = - \frac{E}{R_1 + R_0} + \frac{E(\alpha + 1)}{R_1 + R_0} H\Big(t - \frac{2l}{c} \Big)$$

$$- \frac{E\alpha(\alpha + 1)}{R_1 + R_0} H\Big(t - \frac{4l}{c} \Big) + \ldots, \quad . \quad (32.12)$$

so that the current has a series of constant values, diminishing in magnitude. If $R_0 = R_1 = \sqrt{(L/C)}$, (the characteristic impedance), there is a single pulse of current $- E/2R_0$ for time $2l/c$, and zero current subsequently.

Clearly any terminal impedances can be discussed in the same way, but since the coefficients of the later exponentials are usually rather complicated functions of p, the method is most suitable for use when the solution is needed for fairly small values of the time.

33. *The expansion formula for finite lines*

In chapters I and II the transforms of the solutions of problems on ordinary differential equations were usually found to be of the form

$$\bar{\phi}(p) = \frac{f(p)}{g(p)}, \qquad . \qquad . \qquad (33.1)$$

where $f(p)$ and $g(p)$ were polynomials in p. The solution $\phi(t)$ was found from $\bar{\phi}(p)$ by expressing (33.1) in partial fractions, or by the equivalent technique of Theorem II, for example, by the result (2.8) that if $a_1, \ldots, a_n$ are the zeros of $g(p)$, and these are all different, then

$$\phi(t) = \sum_{r=1}^{n} \frac{f(a_r)}{g'(a_r)} e^{a_r t}. \qquad . \qquad . \qquad (33.2)$$

In the problems on finite lines in this chapter, rather more complicated transforms have appeared. The new feature of these is the occurrence of quotients of hyperbolic functions of q. For example, the expression

$$\bar{\phi}(p) = \frac{\cosh p(l - x)/c}{p \cosh pl/c} \qquad . \qquad . \qquad (33.3)$$

appeared in (31.14). Since $\cosh z$ and $\sinh z$ can be represented by the infinite products *

$$\cosh z = \left(1 + \frac{4z^2}{\pi^2}\right)\left(1 + \frac{4z^2}{3^2\pi^2}\right)\left(1 + \frac{4z^2}{5^2\pi^2}\right) \cdots, \quad (33.4)$$

$$\sinh z = z\left(1 + \frac{z^2}{\pi^2}\right)\left(1 + \frac{z^2}{2^2\pi^2}\right)\left(1 + \frac{z^2}{3^2\pi^2}\right) \cdots, \quad (33.5)$$

* Carslaw, *Plane Trigonometry*, edn. 3, § 168.

8

a quotient of hyperbolic functions such as (33.3) may still be regarded as being of the type

$$\bar{\phi}(p) = \frac{f(p)}{g(p)}, \qquad . \qquad . \qquad . \qquad (33.6)$$

except that now $f(p)$ and $g(p)$ may have an infinite number of factors, instead of a finite number as in (33.1). Then if $a_1, a_2, \ldots$ are the zeros of $g(p)$, and if these are all different, it is plausible to assume that (33.2) can be extended to this case : that is, if $\bar{\phi}(p)$ is given by (33.6),

$$\phi(t) = \sum_{r=1}^{\infty} \frac{f(a_r)}{g'(a_r)} e^{a_r t}. \qquad . \qquad . \qquad (33.7)$$

Just as in § 2 (33.7) applies only to the case in which $g(p)$ has no repeated zeros; if it has, the corresponding extensions (2.19) or (2.20) must be used.

The result (33.7) will be referred to as the *expansion formula ;* it has not been proved, and is merely suggested as a possible extension to the case of an infinite number of zeros, of the result (2.8) which was proved for the case of a finite number of zeros. · That is precisely the approach which was used by Heaviside. The justification for the use of (33.7) is as follows : the rigorous pure mathematical method of finding $\phi(t)$ from (33.6) consists of the application of the inversion theorem (25.2), followed by contour integration and a certain amount of pure mathematical discussion, and, *in the case of finite lines,* * *the final step in this procedure is equivalent to the use of* (33.7). Thus it may be taken that there is a solid theoretical foundation for results on finite lines (or on wave motion, or conduction of heat, in a finite region) obtained by the use of (33.7) ; this remark *does not apply to semi-infinite lines,*† for example (33.7) must not be applied to (30.6).

* Any applied voltages must be combinations of the functions of Table I, and the lines must not be terminated by their characteristic impedance.

† Or to lines terminated by their characteristic impedance— in this case the transforms encountered are of the same type as those for semi-infinite lines.

As a first example of the use of (33.7), we apply it to find $\phi(t)$ from (33.3). First the zeros of the denominator must be found. This could be done by putting * $z = pl/c$ in (33.4), but, from the point of view of the subsequent calculation, the following alternative method is better: clearly $p = 0$ is a zero, and the others are

$$p = \pm \frac{(2n + 1)i\pi c}{2l}, \quad n = 0, \quad 1, \quad 2, \ldots, \quad (33.8)$$

since, for these values of p,

$$\cosh \frac{pl}{c} = \cosh\left\{\frac{(2n + 1)i\pi c}{2l} \cdot \frac{l}{c}\right\} = \cos \frac{(2n + 1)\pi}{2} = 0.$$

To apply (33.7) to (33.3) we need

$$\frac{d}{dp}\left(p \cosh \frac{pl}{c}\right) = \cosh \frac{pl}{c} + \frac{lp}{c} \sinh \frac{pl}{c}. \qquad (33.9)$$

Thus $\qquad \left[\frac{d}{dp}\left(p \cosh \frac{pl}{c}\right)\right]_{p=0} = 1. \qquad . \quad (33.10)$

Also, since $\cosh(pl/c) = 0$ when p has one of the values (33.8), we find from (33.9)

$$\left[\frac{d}{dp}\left(p \cosh \frac{pl}{c}\right)\right]_{p=(2n+1)i\pi c/2l} = \frac{(2n + 1)i\pi}{2} \sinh \frac{(2n + 1)i\pi}{2}$$
$$= \tfrac{1}{2}(-)^{n+1}(2n + 1)\pi. \quad (33.11)$$

Using these results, (33.7) gives from (33.3)

$$\phi(t) = 1 + \sum_{n=0}^{\infty} \frac{2(-)^{n+1}}{(2n+1)\pi}\left\{ \cosh\left[\frac{(2n+1)i\pi}{2}\left(1 - \frac{x}{l}\right)\right]e^{(2n+1)i\pi ct/2l} + \text{Conjugate}\right\}$$

$$= 1 + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-)^{n+1}}{(2n+1)} \cos\left[\frac{(2n+1)\pi}{2}\left(1 - \frac{x}{l}\right)\right] \cos \frac{(2n+1)\pi ct}{2l},$$

$$= 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin \frac{(2n+1)\pi x}{2l} \cos \frac{(2n+1)\pi ct}{2l}, \qquad (33.12)$$

* (33.4) and (33.5) can be used to show the actual factors of numerator and denominator : this is often useful in cases where a factor can be cancelled between them, or where some exceptional factor arises.

where the term 1 has arisen from the zero $p = 0$ of the denominator.

Using this result in (31.14) gives

$$V = \frac{4E}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin \frac{(2n+1)\pi x}{2l} \cos \frac{(2n+1)\pi ct}{2l}. \quad (33.13)$$

A solution of the same problem, expressed in terms of successively reflected waves, was given in (31.15). The connection between this and (33.13) is of course that at any point x, the solution (31.15) is a periodic function of t with period $4l/c$, and if this function is expressed as a Fourier series, the result is (33.13).

As another example of the application of (33.7) we determine V for the first problem of § 31 from its transform (31.6), namely

$$\bar{V} = \frac{E \sinh qx}{p \sinh ql}, \qquad . \qquad . \quad (33.14)$$

considering various forms of q, which was defined in (29.6) by

$$q^2 = (Lp + R)(Cp + G). \qquad . \quad (33.15)$$

The first point to notice is, that if we use the infinite product (33.5) for sinh z in (33.14), we get

$$\bar{V} = \frac{E}{p} \cdot \frac{qx[1 + (q^2x^2/\pi^2)][1 + (q^2x^2/2^2\pi^2)][\ \cdots}{ql[1 + (q^2l^2/\pi^2)][1 + (q^2l^2/2^2\pi^2)][\ \cdots}$$

$$= \frac{E}{p} \cdot \frac{x[1 + (x^2/\pi^2)(Lp+R)(Cp+G)][\ \cdots}{l[1 + (l^2/\pi^2)(Lp+R)(Cp+G)][\ \cdots} \quad (33.16)$$

It follows that although $\bar{V}$, as given in (33.14), appears to be a function of q (which is the square root of a function of p), it in fact involves only q^2, and thus is a function of p which does not involve square roots. The same result holds for all transforms which arise in problems on finite lines.

The general case of (33.14) will be studied in Example 3 below; special cases are given in Examples 1 and 2.

EXAMPLE 1. The "lossless" line, $q = p/c$, where $c = (LC)^{-\frac{1}{2}}$.

In this case (33.14) becomes

$$\bar{V} = \frac{E \sinh px/c}{p \sinh pl/c}. \qquad . \qquad . \quad (33.17)$$

One zero of the denominator is $p = 0$, and the others are

$$p = \pm in\pi c/l, \quad n = 1, \quad 2 \ldots \quad . \quad (33.18)$$

since $\qquad \sinh \left\{ \dfrac{in\pi c}{l} \cdot \dfrac{l}{c} \right\} = i \sin n\pi = 0.$

Also $\quad \left[\dfrac{d}{dp} \left(p \sinh \dfrac{pl}{c} \right) \right]_{p = in\pi c/l} = \left[\dfrac{pl}{c} \cosh \dfrac{pl}{c} \right]_{p = in\pi c/l}$

$$= (-)^n in\pi, \quad n = 1, \quad 2, \ldots \quad (33.19)$$

The zero $p = 0$ of the denominator is clearly exceptional since the numerator of (33.17) also vanishes for this value. The simplest way to see what happens in such a case, and to evaluate the corresponding term in the expansion formula, is to use the series for $\sinh z$ and $\cosh z$, namely

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots, \quad . \quad (33.20)$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^3}{4!} + \ldots, \quad . \quad (33.21)$$

in (33.17), and cancel any common factor. Using (33.20), (33.17) becomes

$$\bar{V} = \frac{E(px/c)[1 + (p^2x^2/6c^2) + \ldots]}{p(pl/c)[1 + (p^2l^2/6c^2) + \ldots]}$$

$$= \frac{Ex[1 + (p^2x^2/6c^2) + \ldots]}{pl[1 + (p^2l^2/6c^2) + \ldots]}, \quad (33.22)$$

and, in this form, the numerator of $\bar{V}$ does not vanish when $p = 0$, and the denominator has only a single zero there.

Also $\quad \left[\dfrac{d}{dp} \left\{ pl \left[1 + \dfrac{p^2l^2}{6c^2} + \ldots \right] \right\} \right]_{=0} = l. \quad . \quad (33.23)$

Using this result for $p = 0$, and (33.19) for the zeros (33.18), we get finally from (33.17) and the expansion formula (33.7)

$$V = \frac{Ex}{l} + \sum_{n=1}^{\infty} \left\{ \frac{(-)^n E}{in\pi} \sinh \frac{in\pi x}{l} e^{in\pi ct/l} + \text{Conjugate} \right\}$$

$$= \frac{Ex}{l} + \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n}{n} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}. \quad (33.24)$$

The minor difficulty which was encountered above at $p = 0$ arises in many problems, and it is always advisable to see precisely what happens at such a point by substituting the series (33.20), (33.21), or the products (33.4), (33.5) for the hyperbolic functions, and cancelling any common factor. In calculating the contribution from such a point, instead of using a term of the type $[f(a_r)e^{a_r t}/g'(a_r)]$ which occurred in (33.7) and (2.8), one of the equivalent type (2.10), namely

$$\left[\frac{(p - a_r)f(p)}{g(p)} \right]_{p=a_r} e^{a_r t} \quad . \quad . \quad (33.25)$$

could have been used. In the problem above, this gives

$$\left[\frac{Ep \sinh px/c}{p \sinh pl/c} \right]_{p=0} = \frac{Ex}{l},$$

which is the first term of (33.24).

Note also that if, after cancelling any common factors as in (33.22), the denominator of $\bar{V}$ has a multiple zero, a term of the type (2.20) must be used for this zero.

EXAMPLE 2.   The submarine cable, $q = (p/\kappa)^{\frac{1}{2}}$, where $\kappa = 1/RC$.

In this case (33.14) becomes

$$\bar{V} = \frac{E \sinh x \sqrt{(p/\kappa)}}{p \sinh l \sqrt{(p/\kappa)}}. \quad . \quad . \quad (33.26)$$

As in Example 1, $p = 0$ is an exceptional value at which both numerator and denominator vanish. Using (33.20) we find

$$\bar{V} = \frac{Ex[1 + (px^2/6\kappa) + \ldots]}{lp[1 + (pl^2/6\kappa) + \ldots]},$$

and $\quad \left[\dfrac{d}{dp}\{lp[1 + (pl^2/6\kappa) + \ldots]\}\right]_{p=0} = l. \quad (33.27)$

The other zeros of the denominator of (33.26) are

$$p = -\kappa n^2\pi^2/l^2, \qquad n = 1, 2, 3 \ldots, \quad (33.28)$$

since $\quad \sinh l\left(-\dfrac{\kappa n^2\pi^2}{\kappa l^2}\right)^{\frac{1}{2}} = \pm \sinh in\pi = 0.$

Also $\quad \left[\dfrac{d}{dp}\left\{p \sinh l\left(\dfrac{p}{\kappa}\right)^{\frac{1}{2}}\right\}\right]_{p=-\kappa n^2\pi^2/l^2}$

$$= \left[\frac{l}{2}\left(\frac{p}{\kappa}\right)^{\frac{1}{2}} \cosh l\left(\frac{p}{\kappa}\right)^{\frac{1}{2}}\right]_{p=-\kappa n^2\pi^2/l^2}. \quad (33.29)$$

Thus we have from (33.26) and (33.7), using (33.27) and (33.29)

$$V = \frac{Ex}{l} + \frac{2E}{l}\sum_{n=1}^{\infty} e^{-\kappa n^2\pi^2 t/l^2}\left[\frac{\sinh x(p/\kappa)^{\frac{1}{2}}}{(p/\kappa)^{\frac{1}{2}} \cosh l(p/\kappa)^{\frac{1}{2}}}\right]_{p=-\kappa n^2\pi^2/l^2}$$

$$= \frac{Ex}{l} + \frac{2E}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n} e^{-\kappa n^2\pi^2 t/l^2} \sin\frac{n\pi x}{l}. \quad . \qquad . \quad (33.30)$$

EXAMPLE 3.   The general case of (33.14) with q given by (33.15). Here $p = 0$ is a zero of the denominator, and

$$\left[\frac{d}{dp}(p \sinh ql)\right]_{p=0} = [\sinh ql]_{p=0} = \sinh l(RG)^{\frac{1}{2}}. \quad (33.31)$$

The other zeros of the denominator are found from the values of q which make $\sinh ql$ zero ; these are $q = 0$ and

$$q = \pm in\pi/l, \quad n = 1, 2 \ldots, \qquad . \quad (33.32)$$

The zero $q = 0$ need not be considered, since a factor q has cancelled between numerator and denominator in (33.16). Using the value (33.15) of q, (33.32) gives

$$(Lp + R)(Cp + G) + \frac{n^2\pi^2}{l^2} = 0, \; n = 1, 2 \ldots, \quad (33.33)$$

This is a quadratic which gives two values of p corresponding to each value of n; it could also have been deduced from (33.16). Using the notation

$$\rho = (R/2L) + (G/2C), \quad \sigma = (R/2L) - (G/2C),$$
$$c = (LC)^{-\frac{1}{2}}, \qquad . \qquad . \qquad (33.34)$$

(33.33) becomes

$$p^2 + 2\rho p + \rho^2 - \sigma^2 + \frac{n^2\pi^2 c^2}{l^2} = 0, \quad n = 1, 2 \quad (33.35)$$

The roots of this are

$$-\rho \pm i\nu_n, \qquad . \qquad . \qquad . \qquad (33.36)$$

where
$$\nu_n = [(n^2\pi^2 c^2/l^2) - \sigma^2]^{\frac{1}{2}}, \quad . \qquad . \qquad (33.37)$$

and we suppose for simplicity that ν_n is real for all values of n, that is $(\pi c/l)^2 > \sigma^2$; if this is not the case (33.35) will have real negative roots for some low values of n, and these must be treated separately.

As usual we need

$$\left[\frac{d}{dp}(p \sinh ql)\right]_{p=-\rho+i\nu_n} = \left[lp\frac{dq}{dp} \cosh ql\right]_{p=-\rho+i\nu_n}$$
$$= \left[\frac{lp(p+\rho)}{qc^2} \cosh ql\right]_{p=-\rho+i\nu_n}$$
$$= \frac{(-)^n l^2(-\rho + i\nu_n)\nu_n}{n\pi c^2} \quad . \quad (33.38)$$

Using (33.31), we have finally by (33.7)

$$V = \frac{E \sinh x(RG)^{\frac{1}{2}}}{\sinh l(RG)^{\frac{1}{2}}} + \frac{\pi E c^2}{l^2} e^{-\rho t}$$
$$\sum_{n=1}^{\infty} \left\{ \frac{(-)^n i n e^{i\nu_n t} \sin(n\pi x/l)}{\nu_n(-\rho + i\nu_n)} + \text{Conj.} \right\}$$
$$= \frac{E \sinh x(RG)^{\frac{1}{2}}}{\sinh l(RG)^{\frac{1}{2}}} + \frac{2\pi E c^2}{l^2} e^{-\rho t}$$
$$\sum_{n=1}^{\infty} \frac{n(-)^n}{\nu_n(\rho^2 + \nu_n^2)^{\frac{1}{2}}} \sin \frac{n\pi x}{l} \cos(\nu_n t - \delta_n),$$

where
$$\tan \delta_n = \rho/\nu_n. \qquad . \qquad . \qquad (33.39)$$

The examples above seem sufficient indication of the method of applying the expansion formula to problems on finite lines. From the results (33.13), (33.24), (33.30), (33.39) it appears that the expansion formula gives the solution as an infinite series, whose terms are products of trigonometric terms in x and exponential or trigonometric terms in t. These are quite different in form to those obtained in §§ 31, 32 in terms of multiply reflected waves. Each type of solution has its own advantages, so that it is hard to give any general indication of which method it is better to use—preferably both should be tried in any specific problem. However, the position as seen from the results of the examples worked out above is as follows :

(i) For the lossless and distortionless lines with simple terminations, both methods are available, but the solution in terms of multiply reflected waves gives the actual wave form of the disturbance with a simple physical interpretaion, and thus is preferable.

(ii) For more general lines the solution by the expansion formula is very little more complicated than that for the lossless line, while the solutions in terms of multiply reflected waves involve much more complicated functions (not given here).

(iii) For lines with terminal impedances, the solution in terms of multiply reflected waves is useful in general only for the first few waves, that is, for not too large values of the time, while the solution obtained by the expansion formula (discussed in § 34) is available for all values of the time, though it is often difficult to determine the zeros of the denominator of its transform.

34. *The application of the expansion formula to the finite line with terminal impedances.*

Problems of this type are treated precisely as in § 33, the only new feature being that the transforms have more complicated denominators whose zeros may be difficult to find. As a specific example we consider a lossless line of length l, insulated at $x = l$, and suppose that at $t = 0$ a

condenser of capacity C_0, charged to voltage E, is discharged into the line at $x = 0$.

The subsidiary equation for the line is, writing $c = (LC)^{-\frac{1}{2}}$,

$$\frac{d^2\bar{V}}{dx^2} - \frac{p^2}{c^2}\bar{V} = 0, \quad 0 < x < l, \qquad (34.1)$$

with

$$\frac{d\bar{V}}{dx} = 0, \quad x = l. \qquad . \qquad . \qquad (34.2)$$

If V_0 and I_0 are the voltage and current at the point $x = 0$ of the line, the subsidiary equation for the condenser C_0 is

$$-\frac{1}{C_0 p}I_0 = \bar{V}_0 - \frac{E}{p}, \qquad . \qquad . \qquad (34.3)$$

that is, using (29.7),

$$\frac{1}{LC_0 p^2}\frac{d\bar{V}}{dx} = \bar{V} - \frac{E}{p}, \quad \text{when } x = 0. \qquad . \quad (34.4)$$

This is the boundary condition to be satisfied at $x = 0$. The solution of (34.1) which satisfies (34.2) is

$$\bar{V} = A \cosh [p(l - x)/c)].$$

Substituting this in (34.4) gives

$$A\left\{\frac{1}{LC_0 c p}\sinh\frac{pl}{c} + \cosh\frac{pl}{c}\right\} = \frac{E}{p}.$$

Thus

$$\bar{V} = \frac{EC_0}{Cc} \cdot \frac{\cosh p(l - x)/c}{\sinh (pl/c) + (pC_0/Cc) \cosh (pl/c)}. \quad (34.5)$$

The zeros of the denominator of (34.5) are $p = 0$, and

$$p = \pm ic\alpha_n/l, \quad n = 1, \quad 2 \ldots, \qquad . \qquad (34.6)$$

where the α_n are the positive roots of

$$\sin \alpha + k\alpha \cos \alpha = 0, \qquad . \qquad . \qquad (34.7)$$

where

$$k = C_0/lC \qquad . \qquad . \qquad . \qquad (34.8)$$

is the ratio of the terminal capacity C_0 to the capacity of

the whole line.   Approximate values of these roots can be found from the intersections of the curves

$$y = \tan \alpha \atop y = -k\alpha \Big\}, $$

and
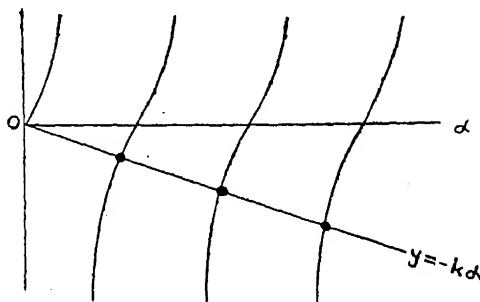
as shown in Fig. 31, and these values can be improved by interpolation.

FIG. 31.

To find V from (34.5) by the expansion formula (33.7), we need

$$\left[\frac{d}{dp} \left\{ \sinh \frac{pl}{c} + \frac{pC_0}{Cc} \cosh \frac{pl}{c} \right\} \right]_{p=ic\alpha_n/l}$$

$$= \left[\left(\frac{C_0}{cC} + \frac{l}{c} \right) \cosh \frac{pl}{c} + \frac{plC_0}{Cc^2} \sinh \frac{pl}{c} \right]_{p=ic\alpha_n/l}$$

$$= \left(\frac{C_0}{cC} + \frac{l}{c} \right) \cos \alpha_n - \frac{C_0\alpha_n}{cC} \sin \alpha_n$$

$$= (l/c)[k + 1 + k^2\alpha_n^2] \cos \alpha_n, \qquad \cdot \qquad \cdot \qquad (34.9)$$

where k is given by (34.8).   Also

$$\left[\frac{d}{dp} \left\{ \sinh \frac{pl}{c} + \frac{pC_0}{Cc} \cosh \frac{pl}{c} \right\} \right]_{p=0} = \frac{l(1+k)}{c}. \quad (34.10)$$

Using (34.9) and (34.10), the expansion formula gives

$$V = \frac{kE}{1 + k} + 2kE \sum_{n=1}^{\infty} \frac{\cos\left[\alpha_n(l - x)/l\right] \cos \alpha_n ct/l}{(k + 1 + k^2 \alpha_n^2) \cos \alpha_n}. \quad (34.11)$$

35. *Further discussion of semi-infinite lines*

In § 30 the transforms of the solutions of problems on semi-infinite lines were found to contain the factor

$$e^{-qx} \qquad . \qquad . \qquad . \qquad (35.1)$$

where $\qquad q = \sqrt{\{(Lp + R)(Cp + G)\}}, \qquad . \qquad (35.2)$

and in §§ 31, 32 the travelling wave solutions for finite lines were found to involve similar factors. It is thus very important to know the functions which correspond to transforms of these types. In the case of the lossless and distortionless lines these can be found by the use of Theorem VI ; in all other cases, either the inversion theorem, or a table of transforms, must be used. Perhaps the most useful collection of transforms is given in Campbell and Foster,* " Fourier Integrals for Practical Applications ", in the relevant part of which, Part 8, most of the results † can be read as Laplace transforms, p being our p, and g being written for our t. Many results for the general case (35.2) are given in their Nos. 860.5 onwards, and, for the submarine cable case, $L = G = 0$, earlier.

Here we consider only the submarine cable case, which corresponds also to linear flow of heat, and give an extension of our table of transforms which will cover a number of problems of this type. The simplest and most important results follow from the known definite integral ‡

$$\int_0^{\infty} e^{-b^2u^2 - (a^2/4u^2)} \, du = \frac{\pi^{\frac{1}{2}}}{2b} e^{-ab}, \qquad . \qquad (35.3)$$

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* Bell System Technical Monograph, B–584. See also McLachlan and Humbert, *Formulaire pour le calcul symbolique* (Gauthier-Villars, 1941).

† Namely, those stated for $g > 0$.

‡ Gibson, *Treatise on the Calculus*, edn. 2, p. 470.

where a and b are positive. Putting $b^2 = p$, and substituting $u = \sqrt{t}$ in this, gives

$$\int_0^\infty e^{-pt - (a^2/4t)} \frac{dt}{t^{\frac{1}{2}}} = \frac{\pi^{\frac{1}{2}}}{p^{\frac{1}{2}}} e^{-ap^{\frac{1}{2}}}. \qquad (35.4)$$

Integrating (35.4) with respect to the parameter* a, from a to ∞, gives

$$\int_0^\infty e^{-pt} t^{-\frac{1}{2}} dt \int_a^\infty e^{-a^2/4t} da = \frac{\pi^{\frac{1}{2}}}{p^{\frac{1}{2}}} \int_a^\infty e^{-ap^{\frac{1}{2}}} da \quad (35.5)$$

Writing

$$\operatorname{erf} x = \frac{2}{\pi^{\frac{1}{2}}} \int_0^x e^{-u^2} du, \qquad . \qquad (35.6)$$

$$\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\pi^{\frac{1}{2}}} \int_x^\infty e^{-u^2} du, \quad . \quad (35.7)$$

for the error function (numerical values of $\operatorname{erf} x$ are given in most books of tables) (35.5) becomes

$$\int_0^\infty e^{-pt} \operatorname{erfc} \left\{ \frac{a}{2t^{\frac{1}{2}}} \right\} dt = \frac{e^{-ap^{\frac{1}{2}}}}{p}. \qquad (35.8)$$

In Table 2 the results (35.4) and (35.8) are given, together with some others which are useful in more complicated problems. Other transforms can be deduced from these by Theorems I, III, IV, IX, also by differentiating or integrating with respect to a parameter, just as (35.8) was deduced from (35.4). For example (35.9) follows from (35.10) by differentiating with respect to the parameter a (or alternatively it may be derived independently from (35.3) by putting $u = t^{-\frac{1}{2}}$). Also (35.11) may be deduced from (35.9) by Theorem IV. The results (35.12) to (35.14) may be proved by a rather more elaborate discussion of the same type.

* The processes of differentiating or integrating under the integral sign with respect to a parameter are most useful, since they often allow a chain of transforms (such as (35.9) to (35.11), or (35.12) to (35.14)) to be deduced from a single transform. It must always be understood that the use of these processes needs justification, but this is usually readily supplied for integrals of the types considered here.

Finally, the Laplace transform of t^ν, where ν is not restricted to be a positive integer as in Table I, must be mentioned. This is

$$\int_0^\infty e^{-pt}t^\nu\,dt = \frac{1}{p^{\nu+1}}\int_0^\infty e^{-\xi}\xi^\nu d\xi \qquad . \quad (35.16)$$

TABLE 2

In this Table a and b are positive or zero

$\bar{y}(p)$.	$y(t)$.
$e^{-ap^{\frac{1}{2}}}$	$\frac{1}{2}a(\pi t^3)^{-\frac{1}{2}}e^{-a^2/4t}$ . . (35.9)
$p^{-\frac{1}{2}}e^{-ap^{\frac{1}{2}}}$	$(\pi t)^{-\frac{1}{2}}e^{-a^2/4t}$ . . (35.10)
$p^{-1}e^{-ap^{\frac{1}{2}}}$	$\mathrm{erfc}\,(\frac{1}{2}at^{-\frac{1}{2}})$ . . (35.11)
$\dfrac{e^{-ap^{\frac{1}{2}}}}{b + p^{\frac{1}{2}}}$	$(\pi t)^{-\frac{1}{2}}e^{-a^2/4t} - be^{ab+b^2t}\,\mathrm{erfc}\left(\dfrac{a}{2t^{\frac{1}{2}}} + bt^{\frac{1}{2}}\right)$ (35.12)
$\dfrac{e^{-ap^{\frac{1}{2}}}}{p + bp^{\frac{1}{2}}}$	$e^{ab+b^2t}\,\mathrm{erfc}\left(\dfrac{a}{2t^{\frac{1}{2}}} + bt^{\frac{1}{2}}\right)$. (35.13)
$\dfrac{e^{-ap^{\frac{1}{2}}}}{p(b + p^{\frac{1}{2}})}$	$\dfrac{1}{b}\,\mathrm{erfc}\left(\dfrac{a}{2t^{\frac{1}{2}}}\right) - \dfrac{1}{b}\,e^{ab+b^2t}\,\mathrm{erfc}\left(\dfrac{a}{2t^{\frac{1}{2}}} + bt^{\frac{1}{2}}\right)$ (35.14)
$\dfrac{1}{p^\nu}$	$\dfrac{t^{\nu-1}}{\Gamma(\nu)},\quad \nu > 0$ . . (35.15)

If $\nu > -1$, the integral in (35.16) exists and is denoted by $\Gamma(\nu + 1)$; this gives the result (35.15). Numerical values of the Γ function are given in most books of tables. Its most important properties are

$$\Gamma(\nu + 1) = \nu\Gamma(\nu),$$

and $\Gamma(n) = (n - 1)!$, if n is a positive integer,   (35.17) so that (1.5) is a special case of (35.15), as it should be. Also

$$\Gamma(\tfrac{1}{2}) = \pi^{\frac{1}{2}}, \text{ and } \Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\pi^{\frac{1}{2}}, \qquad . \quad (35.18)$$

so that the special case $a = 0$ of (35.10) agrees with (35.15).

As examples of the use of Table 2 we consider the following problems on the semi-infinite submarine cable.

EXAMPLE 1.   Constant voltage E applied at $x = 0$ at $t = 0$ with zero initial conditions.   Here, from (30.6) with $L = G = 0$,

$$\bar{V} = \frac{E}{p} e^{-x(p/\kappa)^{\frac{1}{2}}} \qquad . \qquad . \quad (35.19)$$

where

$$\kappa = 1/RC. \qquad . \qquad . \qquad (35.20)$$

Then it follows from (35.11) that

$$V = E \, \text{erfc} \left\{ \frac{x}{2(\kappa t)^{\frac{1}{2}}} \right\} \qquad . \qquad . \quad (35.21)$$

Using (29.7), the current I is to be found from

$$I = -\frac{1}{R} \frac{d\bar{V}}{dx} = \frac{EC^{\frac{1}{2}}}{(Rp)^{\frac{1}{2}}} e^{-x(p/\kappa)^{\frac{1}{2}}} \qquad . \quad (35.22)$$

Thus, from (35.10),

$$I = \frac{EC^{\frac{1}{2}}}{(\pi R t)^{\frac{1}{2}}} e^{-x^2/4\kappa t} \qquad . \qquad . \quad (35.23)$$

If we had only required the current I_0 at $x = 0$, (35.22) becomes in this case

$$\bar{I}_0 = \frac{EC^{\frac{1}{2}}}{(Rp)^{\frac{1}{2}}} \qquad . \qquad . \qquad . \quad (35.24)$$

and, from (35.15) and (35.18),

$$I_0 = \frac{EC^{\frac{1}{2}}}{(\pi R t)^{\frac{1}{2}}} \qquad . \qquad . \qquad . \quad (35.25)$$

EXAMPLE 2.   The problem of Example 1, except that the voltage is applied to the line through an impedance z_0.

As in § 30, the subsidiary equation for the line is

$$\frac{d^2\bar{V}}{dx^2} - \frac{p}{\kappa} \bar{V} = 0, \; x > 0, \; . \qquad . \quad (35.26)$$

where κ is defined in (35.20).   The solution of (35.26) which remains finite as $x \to \infty$ is

$$\bar{V} = Ae^{-x\sqrt{(p/\kappa)}}. \qquad . \qquad . \quad (35.27)$$

Also, by (29.7),

$$\bar{I} = -\frac{1}{R}\frac{d\bar{V}}{dx} = \frac{Ap^{\frac{1}{2}}}{R\kappa^{\frac{1}{2}}}e^{-x\sqrt{(p/\kappa)}} \qquad . \quad (35.28)$$

The constant A is found from the boundary condition (32.2),

$$z_0\bar{I} = \frac{E}{p} - \bar{V}, \; x = 0.$$

That is, $A\left\{\dfrac{z_0p^{\frac{1}{2}}}{R\kappa^{\frac{1}{2}}} + 1\right\} = \dfrac{E}{p}.$

Thus $\bar{V} = \dfrac{ER\kappa^{\frac{1}{2}}}{p\{z_0p^{\frac{1}{2}} + R\kappa^{\frac{1}{2}}\}}e^{-x\sqrt{(p/\kappa)}} \qquad . \quad (35.29)$

If z_0 is a resistance, R_0, it follows from (35.14) that

$$V = E\,\text{erfc}\,\frac{x}{2\sqrt{(\kappa t)}} - Ee^{kx+k^2\kappa t}\,\text{erfc}\left(\frac{x}{2\sqrt{(\kappa t)}} + k\sqrt{(\kappa t)}\right),$$
$$(35.30)$$

where $k = R/R_0$.

Again, if z_0 is a condenser of capacity C_0, (35.13) gives

$$V = E\,exp\left\{\frac{t}{R^2C_0{}^2\kappa} + \frac{x}{RC_0\kappa}\right\}\,\text{erfc}\left\{\frac{x}{2\sqrt{(\kappa t)}} + \frac{t^{\frac{1}{2}}}{RC_0\kappa^{\frac{1}{2}}}\right\}$$
$$(35.31)$$

For other values of z_0, the expression $1/(z_0p^{\frac{1}{2}} + R\kappa^{\frac{1}{2}})$ in (35.29) must be expressed in partial fractions whose denominators are linear in $p^{\frac{1}{2}}$.

EXAMPLES ON CHAPTER IV

1. Voltage $E\sin(\omega t + \alpha)$ is applied at $t = 0$ through a resistance R_0 to a semi-infinite lossless line $x > 0$, with zero initial current and charge. Show that the voltage in the line is

$$\frac{ER_1}{R_0 + R_1}\sin\left[\omega\left(t - \frac{x}{c}\right) + \alpha\right]H\left(t - \frac{x}{c}\right),$$

where $c = (LC)^{-\frac{1}{2}}$, $R_1 = \sqrt{(L/C)}$.

2. Constant voltage E is applied at $t = 0$ at the end $x = 0$ of a lossless line of length l, the end $x = l$ being insulated. If

the initial current and charge are zero, show that the voltage at any point of the line is

$$E \sum_{n=0}^{\infty} (-1)^n \left\{ H\left(t - \frac{2nl + x}{c}\right) + H\left(t - \frac{(2n + 2)l - x}{c}\right) \right\}.$$

3. A lossless transmission line extends from $x = -l$ to $x = a$, the end $x = a$ being insulated, and the end $x = -l$ being earthed. At $t = 0$ there is no current in the line, the portion $0 < x < a$ is charged to voltage E, and the portion $-l < x < 0$ is uncharged. Show that the current at $x = -l$ at time t is

$$-E \sqrt{\left(\frac{C}{L}\right)} \sum_{n=0}^{\infty} (-1)^n \left\{ H\left[t - \frac{2n(l + a) + l}{c}\right] \right. $$
$$\left. - H\left[t - \frac{(2n + 1)(l + a) + a}{c}\right] \right\},$$

where $c = (LC)^{-\frac{1}{2}}$.

4. Constant voltage E is applied at $t = 0$ through an inductive resistance L_0, R_0 to the end $x = 0$ of a lossless line of length l, the end $x = l$ being insulated, and there being zero initial current and charge. Show that the current at $x = 0$ is

$$\frac{E}{R_0 + R_1}(1 - e^{-\alpha t})$$
$$- \frac{2ER_1}{(R_0 + R_1)^2} \left\{ 1 - \left[\alpha t + 1 - \frac{2\alpha l}{c}\right] e^{-\alpha(t - 2l/c)} \right\} H\left(t - \frac{2l}{c}\right) + \ldots,$$

where $c = (LC)^{-\frac{1}{2}}$, $R_1 = (L/C)^{\frac{1}{2}}$, and $\alpha = (R_0 + R_1/L_0$.

5. A submarine cable ($L = G = 0$, and $\kappa = 1/RC$ in (29.10)) of length l has zero initial current and charge. The end $x = 0$ is insulated, and a constant voltage E is applied at $x = l$, show that the voltage at any point is

$$E + \frac{4E}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n - 1)} e^{-\kappa(2n-1)^2 \pi^2 t/4l^2} \cos \frac{(2n - 1)\pi x}{2l}.$$

6. A lossless line of length l has zero initial current and charge, and is insulated at $x = l$. Voltage $E \sin \omega t$ is applied at $x = 0$ for $t > 0$, show that the voltage at any point is

$$\frac{E \cos [\omega(l - x)/c] \sin \omega t}{\cos \omega l/c}$$
$$+ 8E\omega lc \sum_{n=0}^{\infty} \frac{\sin [(2n + 1)\pi ct/2l] \sin [(2n + 1)\pi x/2l]}{4\omega^2 l^2 - (2n + 1)^2 \pi^2 c^2},$$

9

where $c = (LC)^{-\frac{1}{2}}$, and ω is not to be equal to $(2n + 1)\pi c/2l$ for any n.

7. A submarine cable of length l has the end $x = l$ earthed, and constant voltage E is applied at $x = 0$ with zero initial conditions. Show, using the method of § 31, that the current at $x = 0$ is

$$\frac{E}{R\sqrt{(\pi \kappa t)}}\left\{1 + 2\sum_{n=1}^{\infty} e^{-n^2 l^2/\kappa t}\right\}.$$

INDEX